Time-reversal frameness and superselection

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We show that appropriate superpositions of motional states are a reference frame resource that enables breaking of time-reversal superselection so that two parties lacking knowledge about the other's direction of time can still communicate. We identify the time-reversal reference frame resource states and determine the corresponding frameness monotone, which connects time-reversal frameness to entanglement. In contradistinction to other studies of reference frame quantum resources, this is the first analysis that involves an antiunitary rather than unitary representation. © *2009 American Institute of Physics.* [doi:10.1063/1.3243821]

I. INTRODUCTION

Superselection in quantum mechanics[1] is the principle that certain types of coherent superpositions are forbidden, and operations on Hilbert space must be invariant under group transformations that would allow violation of the superselection principle. At times, superselection has been touted as a fundamental principle to prevent such counterintuitive phenomena such as superpositions of different charges that are otherwise permissible in quantum mechanics. More recently superselection has been given an informational interpretation that is associated with lack of a reference frame[2,3] such as orientation, chirality, or phase.[4] In the cases studied so far, the superselection is separated from coherence by ignorance or knowledge of which unitary transformation to apply in a particular frame, and this knowledge is provided by a reference frame that can be quantum or classical in nature. In the quantum case, the reference frame is consumed through its use to measure quantities that would otherwise be forbidden by superselection.[5,6]

The relationship between reference frames and superselection bears similarity to entanglement theory and restrictions under local operations and classical communication (LOCC).[7] Moreover, entanglement as a resource to break LOCC restrictions is related to reference frames as a resource for violating superselection. The reference frame resource is known as *frameness*, and there are corresponding monotones analogous to entanglement monotones; these frameness monotones are nondecreasing under *G*-invariant operations, where *G* is the group associated with the superselection rule.

Here we consider the problem of time-reversal (TR) invariance. Consider two parties, Alice and Bob, who wish to communicate but do not know which direction in time applies to the other party. In an exotic setting, Alice and Bob could be both moving forward or backward in time or one could be moving forward and the other backwards in time. In a more practical situation, Alice and Bob could be unsure whether their directions of motion are coaligned or counteraligned, as TR corresponds to motion reversal.

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The lack of knowledge about direction of time yields a TR superselection rule that can be broken by Alice and Bob sharing a TR resource. Whereas previous studies of resources to break superselection rules have all involved invariance under unitary group actions, TR is novel in that the corresponding transformations are antiunitary. Here we introduce the notion of a TR-invariant operation (TRIO) and corresponding TRIO equivalence classes develop a resource theory for TR, establish three types of TRIO frameness monotones analogous to the entanglement monotone types "deterministic,"[8] "ensemble,"[9] and "stochastic,"[10] and we propose TR frameness distillation analogous to entanglement distillation.[11]

II. REPRESENTATION OF TR

Classical mechanics is TR invariant: the discrete transformation of extended phase space, **T**, that fixes positions and reverses both momenta and time,

$$\mathbf{T}: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} : (\boldsymbol{x}, \boldsymbol{p}, t) \mapsto (\boldsymbol{x}, -\boldsymbol{p}, -t), \tag{2.1}$$

does not change Hamilton's equations of motion for TR symmetric Hamiltonians. Note that classical angular momentum, $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$, is also reversed under this transformation. It is clear that two successive TRs give the identity transformation, $\mathbf{T}^2 = \mathbf{I}$. Thus, $\{\mathbf{T}, \mathbf{I}\} \cong S_2$, which is the symmetric group of two elements.

In quantum mechanics, TR must be a transformation of Hilbert space: we denote a representation of S_2 carried by bounded operators on Hilbert space by Θ,

$$\Theta: S_2 \to \mathcal{B}(\mathcal{H}): \begin{cases} \mathbf{T} \mapsto \hat{\theta} \\ \mathbf{I} \mapsto \mathbb{1} \end{cases}. \tag{2.2}$$

Following Wigner,[12] if quantum mechanics is also to be TR invariant, then the commutation relations $[\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}] = i$ must be left invariant by $\hat{\theta}$, which gives

$$\hat{\theta}[\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}]\hat{\theta}^{-1} = [\hat{\boldsymbol{x}}, -\hat{\boldsymbol{p}}] = -i = \hat{\theta} i \hat{\theta}^{-1}. \tag{2.3}$$

This implies that $\hat{\theta}$ must be antilinear, equivalent to a linear operator composed with the complex conjugation operator.[13] Since complex conjugation must be performed with respect to some basis, $\hat{\theta}$ is basis dependent and the antilinearity is given by

$$\hat{\theta}(a|\psi\rangle + b|\phi\rangle) = a^*\hat{\theta}|\psi\rangle + b^*\hat{\theta}|\phi\rangle, \quad a, b \in \mathbb{C}, \tag{2.4}$$

for $|\psi\rangle, |\phi\rangle$ in this basis.

Following Sakurai[14] and Landau and Lifshitz[15] we choose as our basis states those of multiple orbital angular momenta,

$$\mathcal{H} = \text{span}\{|\mu\ell m\rangle\}, \quad \langle\mu\ell m|\mu'\ell'm'\rangle = \delta_{\mu\mu'}\delta_{\ell\ell'}\delta_{mm'}, \tag{2.5}$$

where $\ell \in \mathbb{N}$ labels the angular momentum with

$$\text{spec}(\hat{\boldsymbol{L}}^2) = \{\ell(\ell+1)\}, \quad m \in \text{spec}(\hat{L}_z) = \{-\ell, -\ell+1, \dots, \ell\}, \tag{2.6}$$

and $\mu \in \mathbb{Z}^+$ is a multiplicity label distinguishing orthogonal states with the same angular momentum. Interpreting m as the projection of a classical total angular momentum vector onto the z-axis, and since classical angular momentum is reversed by TR, suggests that the TR operator $\hat{\theta}$ should send states with quantum number m to those with quantum number $-m$. Including a state (but not multiplicity-)-dependent phase gives the TR action,

$$\hat{\theta}|\mu\ell m\rangle = e^{i\theta_{\ell m}}|\mu\ell - m\rangle. \tag{2.7}$$

Remark: TR Θ does not mix states of different ℓ; hence the SO(3) irreps labeled by ℓ are intact under TR.

The phase $\theta_{\ell m}$ in Eq. (2.7) is not unique, although it must vary as $\pm \pi m$, and is given differently by several authors. For example, Sakurai[14] gave $\theta_{\ell m} = \pi m$, whereas Landau and Lifshitz[15] employed

$$\theta_{\ell m} = \pi(\ell - m). \tag{2.8}$$

Either convention is satisfactory in this section, but later we prefer the Landau and Lifshitz convention as it respects the Clebsch–Gordan decomposition for two particles, for which the choice $\theta_{\ell m} = \pi(\ell - m)$ yields

$$
\begin{aligned}
\hat{\theta}(|\ell m\rangle|\ell' m'\rangle) &= \hat{\theta}\sum_L (\ell m; \ell' m'|LM)|LM\rangle \\
&= \sum_L (\ell m; \ell' m'|LM)^* e^{i\pi(L-M)}|L-M\rangle \\
&= (-1)^{\ell+\ell'+M'}\sum_L (\ell - m; \ell' - m'|LM')|LM'\rangle \\
&= \hat{\theta}|\ell m\rangle \otimes \hat{\theta}|\ell' m'\rangle,
\end{aligned}
\tag{2.9}
$$

where we have used real Clebsch–Gordan coefficients, the dummy variable $M' = -m - m'$, and the identity

$$(\ell m; \ell' m'|LM) = (-1)^{\ell+\ell'-L}(\ell - m; \ell' - m'|L-M). \tag{2.10}$$

For the choice $\theta_{\ell m} = \pi(\ell - m)$, we have

$$\hat{\theta}(|\psi\rangle \otimes |\phi\rangle) = \hat{\theta}|\psi\rangle \otimes \hat{\theta}|\phi\rangle, \tag{2.11}$$

which defines TR for multiple systems.

Although the basis $\{|\mu lm\rangle\}$ is motivated by physics, for the purpose of quantifying the resources, we will use a different basis that is mathematically much more natural to work with. That is, the *self-conjugate* basis. According to Lemma 1 in Ref. 16 for any conjugation like the TR operation $\hat{\theta}$, there exists an orthonormal basis $\{|e_n\rangle\}$, called the self-conjugate basis for which $\hat{\theta}|e_n\rangle = |e_n\rangle$. In the following lemma, we express this self-conjugate basis in terms of the basis $\{|\mu lm\rangle\}$.

Lemma 1: The self-conjugate basis $|e_n\rangle$, where n stands for the set $\{\mu lm\pm\}$, is given by

$$|e_{\mu \ell 0+}\rangle := e^{i\theta_{\ell 0}/2}|\mu\ell 0\rangle, \quad \ell \geq 0, \quad m = 0,$$

$$|e_{\mu \ell m+}\rangle := \frac{1}{\sqrt{2}}(|\mu\ell m\rangle + e^{i\theta_{\ell m}}|\mu\ell - m\rangle), \quad 0 < m \leq \ell \neq 0,$$

$$|e_{\mu \ell m-}\rangle := \frac{i}{\sqrt{2}}(-|\mu\ell m\rangle + e^{i\theta_{\ell m}}|\mu\ell - m\rangle), \quad 0 < m \leq \ell \neq 0. \tag{2.12}$$

Proof: Consider an arbitrary state expanded in the original basis (2.5),

$$|\psi\rangle = \sum_{\mu\ell m} \psi_{\mu\ell m}|\mu\ell m\rangle \in \mathcal{H}. \tag{2.13}$$

The left-hand side of the eigenequation,

$$\hat{\theta}|\psi\rangle = |\psi\rangle,\tag{2.14}$$

yields

$$\hat{\theta}|\psi\rangle = \hat{\theta}\sum_{\mu\ell m}\psi_{\mu\ell m}|\mu\ell m\rangle = \sum_{\mu\ell m}\psi_{\mu\ell m}^* e^{i\theta_{\ell m}}|\mu\ell -m\rangle = \sum_{\mu\ell m}|\psi_{\mu\ell m}|\exp[-i\arg\psi_{\mu\ell m}+i\theta_{\ell m}]|\mu\ell -m\rangle,\tag{2.15}$$

and the right-hand side of (2.14) gives

$$|\psi\rangle = \sum_{\mu\ell m}\psi_{\mu\ell m}|\mu\ell m\rangle = \sum_{\mu\ell m}\psi_{\mu\ell -m}|\mu\ell -m\rangle = \sum_{\mu\ell m}|\psi_{\mu\ell -m}|\exp[i\arg\psi_{\mu\ell -m}]|\mu\ell -m\rangle.\tag{2.16}$$

As the bases coincide, real and imaginary coefficients can be equated. Normalization implies

$$|\psi_{\mu\ell -m}| = |\psi_{\mu\ell m}|.\tag{2.17}$$

The imaginary part implies that

$$\arg\psi_{\mu\ell m} + \arg\psi_{\mu\ell -m} = \theta_{\ell m}\bmod 2\pi.\tag{2.18}$$

Notice that the left-hand side is symmetric under a change in the sign of m, which is true for integer m when $\theta_{\ell m}$ varies as $\pm\pi m$.

From conditions (2.17) and (2.18), the eigenvectors of $\hat{\theta}$ are given by Eq. (2.12). $\square$

Remark: If one were to consider spinor representations of SO(3), then both ℓ and m would be half-odd integer. In this case the condition on m in (2.18) fails, which implies that every spinor state is a TR resource.

Definition 1: A *TR-invariant pure state* is a vector $|\psi\rangle \in \mathcal{H}$ whose corresponding projection is equal to its TR group average,

$$|\psi\rangle\langle\psi| = \frac{1}{|S_2|}\sum_{g\in S_2}\Theta(g)|\psi\rangle\langle\psi|\Theta(g)^\dagger = \frac{1}{2}\mathbb{1}|\psi\rangle\langle\psi|\mathbb{1} + \frac{1}{2}\hat{\theta}|\psi\rangle\langle\psi|\hat{\theta}^\dagger = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\widetilde{\psi}\rangle\langle\widetilde{\psi}|\tag{2.19}$$

for $|\widetilde{\psi}\rangle := \hat{\theta}|\psi\rangle$.

A state $|\psi\rangle$ is a TR resource if it is not invariant under TR, as it then must contain some information about time's "arrow." Thus, the states $|\mu\ell m\rangle$ are resources as they are not TR invariant. To begin studying TR resources, we first identify zero-resource states. From (2.19), we see that these are necessarily eigenstates of $\hat{\theta}$, and must satisfy $\hat{\theta}|\psi\rangle = e^{i\phi}|\psi\rangle$. Hence, the states $|e_n\rangle$ are nonresource states. Working with the self-conjugate basis (2.12) simplifies the behavior of pure states under TR,

$$\hat{\theta}\sum_n\psi_n|e_n\rangle = \sum_n\psi_n^*|e_n\rangle.\tag{2.20}$$

This leads us to the complete characterization of nonresource states.

Lemma 2: The state $|\psi\rangle$ is TR invariant state (i.e., nonresource state) if and only if $\psi_n = \langle e_n|\psi\rangle$ is real for all n.

Proof: A state $|\psi\rangle$ is TR invariant if and only if $|\psi\rangle = e^{i\phi}\hat{\theta}|\psi\rangle$. In the self-conjugate basis of Eq. (2.20), this implies $\psi_n = e^{i\phi}\psi_n^*$. Letting $\psi_n = |\psi_n|e^{i\phi_n}$, we must have $\phi_n = \phi/2\bmod\pi$. So $\psi_n = |\psi_n|e^{i\phi/2}$, which is real up to an unimportant n-independent global phase. $\square$

III. TR INVARIANT OPERATIONS

If two parties, Alice and Bob, lack a shared reference frame that informs each party of the other's direction of time, their descriptions of the other's operators require averaging over the group action that relates their frames. Schur's lemma tells us that this averaging results in group invariant (or covariant) operators.[3] In this section we classify the completely positive (CP) maps that are TRIOs.

A. Efficient maps

We begin with a study of efficient maps.

Definition 2: An *efficient map* is a map whose Kraus decomposition comprises only one Kraus operator.

Unitary (reversible) operations are examples of efficient CP maps. The Kraus operator can be expressed in the $|e_n\rangle$ representation as

$$K = \sum_{nn'} K_{nn'}|e_n\rangle\langle e'_n|. \tag{3.1}$$

The superoperator,

$$\mathcal{K}(\bullet) = K \bullet K^\dagger, \tag{3.2}$$

acting on a state projector $\bullet \rightarrow |\psi\rangle\langle\psi|$ is a TRIO if

$$K\hat{\theta}|\psi\rangle\langle\psi|\hat{\theta}^\dagger K^\dagger = \hat{\theta}K|\psi\rangle\langle\psi|K^\dagger\hat{\theta}^\dagger \ \forall \ |\psi\rangle \in \mathcal{H}. \tag{3.3}$$

Theorem 3: *An efficient map $\mathcal{K}(\bullet) = K\bullet K^\dagger$ is TRIO if and only if $K_{nn'}$ are all real (that is, K is real in the self-conjugate basis).*

Remark: Note that a constant global phase φ can always be eliminated because

$$\mathcal{K}(\bullet) = K \bullet K^\dagger = (e^{i\varphi}K)(\bullet)(e^{i\varphi}K)^\dagger. \tag{3.4}$$

Proof: Working with the self-conjugate basis, $K|\psi\rangle$ is real for any real $|\psi\rangle$ if and only if K is itself real. $\qquad\square$

B. Standard resources

In this subsection we show that all TRIO resource states can be parametrized by a single real variable θ.

Theorem 4: *For all $|\psi\rangle \in \mathcal{H} \ \exists$ a unitary TRIO Kraus operator K, such that*

$$K|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\theta} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \quad 0 \le \theta \le \frac{\pi}{2}. \tag{3.5}$$

Proof: A unitary TRIO K is real in the self-conjugate basis and therefore orthogonal. Any state in this basis can be written as

$$|\psi\rangle = \sum_n \psi_n|e_n\rangle = \sum_n (\psi_n^R + i\psi_n^J)|e_n\rangle, \tag{3.6}$$

where the real numbers ψ_n^R and ψ_n^J define real vectors ψ^R and ψ^J. Let $a = \|\psi^R\|$, and define $x = \psi^R/a$, a normalized real vector. Further, ψ^J can be decomposed as $\psi^J = bx + cy$, where y is a

normalized real vector orthogonal to x. Thus we have $\psi = (a+ib)x + icy$, with a normalization condition $a^2 + b^2 + c^2 = 1$. Now let x and y be the two first columns of a unitary K', then $\psi = K'(a+ib, ic, 0, \ldots, 0)^T$.

By a further 2×2 rotation R_α, governed by a single rotation angle α, ψ can be brought to the form $|\psi\rangle = e^{i\gamma} K' R_\alpha (1, e^{i\theta}, 0, \ldots, 0)^T \equiv K(1, e^{i\theta}, 0, \ldots, 0)^T$, where γ is an irrelevant global phase. To find α one has to look at the Bloch sphere representations of the density matrices of $(a+ib, ic)^T$ and $(1, e^{i\theta})^T / \sqrt{2}$. Since both are pure states, they lie on the surface. The latter state is a point in the equatorial plane. As a real rotation has the effect of rotating along an axis in that same plane (to wit, the axis corresponding to the σ_y Pauli matrix, which is left invariant under real rotations), any point on the sphere can be covered. $\qquad\square$

Theorem 4 shows that all TRIO resources are quantified by a single real variable θ.

C. Monotones and interconvertibility

From Theorem 4, every resource can be characterized by a qubit state of the form

$$|\psi\rangle = \frac{e^{i\theta/2}}{\sqrt{2}}(|0\rangle + e^{-i\theta}|1\rangle) = \frac{1}{\sqrt{2}}(e^{i\theta/2}|0\rangle + e^{-i\theta/2}|1\rangle), \qquad (3.7)$$

where $0 \leq \theta \leq \pi/2$. We refer to expression (3.7) as the standard representation of the resource. From Theorem 3, the allowed Kraus operators (i.e., TRIO invariant efficient operations) are therefore 2×2 *real* matrices, and the following definition of a monotone is thus reasonable.

Definition 3: For $|\psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle$ with $|\psi_0|^2 + |\psi_1|^2 = 1$, the *TR monotone* $\tau(|\psi\rangle)$ is

$$\tau(|\psi\rangle) = 1 - |\langle \psi^* | \psi \rangle| = 1 - |\psi_0^2 + \psi_1^2| = 1 - \cos\theta. \qquad (3.8)$$

Theorem 5: *The TR monotone τ is an ensemble monotone under TRIO.*

Proof: Consider a measurement represented by TRIO Kraus operators K_k that transforms the initial pure state to the ensemble,

$$|\psi\rangle \mapsto \left\{ p_k, |\varphi_k\rangle \equiv \frac{1}{\sqrt{p_k}} K_k |\psi\rangle \right\}, \qquad (3.9)$$

where p_k is the probability of each outcome. Thus,

$$\sum_k p_k \tau(|\varphi_k\rangle) = \sum_k p_k (1 - |\langle \varphi_k^* | \varphi_k \rangle|) = 1 - \sum_k p_k |\langle \varphi_k^* | \varphi_k \rangle| = 1 - \sum_k |\langle (K_k \psi)^* | K_k \psi \rangle|, \qquad (3.10)$$

which leads to the inequality

$$\sum_k p_k \tau(|\varphi_k\rangle) \leq 1 - \left| \sum_k \langle (K_k \psi)^* | K_k \psi \rangle \right| = 1 - \left| \sum_k \langle \psi^* | K_k^\dagger K_k | \psi \rangle \right| = 1 - |\langle \psi^* | \psi \rangle| = \tau(|\psi\rangle),$$

$$(3.11)$$

where we have used the fact that K_k are real matrices and $\Sigma_k K_k^\dagger K_k = \mathbb{1}$. $\qquad\square$

Theorem 6: *(Analog of Nielsen's theorem[8]) The transformation $|\psi\rangle \rightarrow |\varphi\rangle$ can be achieved deterministically by TRIO if and only if $\tau(|\psi\rangle) \geq \tau(|\varphi\rangle)$.*

Proof: As τ is a monotone, the transformation is achievable by TRIO provided that the condition $\tau(|\psi\rangle) \geq \tau(|\varphi\rangle)$ is met. Thus the condition is necessary, and now we have to show that the condition is also sufficient.

To show sufficiency, observe that, without loss of generality, $|\psi\rangle$ and $|\varphi\rangle$ are expressed in our standard form,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(e^{i\theta/2}|0\rangle + e^{-i\theta/2}|1\rangle) \tag{3.12}$$

and

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(e^{i\gamma/2}|0\rangle + e^{-i\gamma/2}|1\rangle), \tag{3.13}$$

where $\theta, \gamma \in [0, \pi/2]$. The condition $\tau(|\psi\rangle) \geq \tau(|\varphi\rangle)$ is therefore equivalent to $\cos\gamma \geq \cos\theta$ (or $\gamma \leq \theta$). We define a measurement in terms of its two Kraus operators,

$$\hat{K}_1 = \begin{bmatrix} \sqrt{A/2} & \sqrt{(1-A)/2} \\ \sqrt{A/2} & -\sqrt{(1-A)/2} \end{bmatrix} \tag{3.14}$$

and

$$\hat{K}_2 = \begin{bmatrix} -\sqrt{(1-A)/2} & \sqrt{A/2} \\ \sqrt{(1-A)/2} & \sqrt{A/2} \end{bmatrix}, \tag{3.15}$$

with

$$A = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{\cos^2\gamma - \cos^2\theta}{1 - \cos^2\theta}}. \tag{3.16}$$

(Note that $\theta = 0 \Rightarrow \gamma = 0$ and the problem is trivial.) As the Kraus operators are real, the measurement is TRIO.

By Luder's update rule, subsequent to the measurement, for this choice of the parameter A, the state is either

$$|\varphi_1\rangle = \sqrt{2}\hat{K}_1|\psi\rangle \tag{3.17}$$

or

$$|\varphi_2\rangle = \sqrt{2}\hat{K}_2|\psi\rangle, \tag{3.18}$$

with

$$\tau(|\varphi_1\rangle) = \tau(|\varphi_2\rangle) = \tau(|\varphi\rangle).$$

Thus, $|\varphi_1\rangle$, $|\varphi_2\rangle$, and $|\varphi\rangle$ are equivalent up to an orthogonal matrix corresponding to a reversible TRIO. □

Theorem 7: (*Analog of Jonathan–Plenio theorem*[9]) *Every transformation* $\mathcal{T}:|\psi\rangle \rightarrow \{p_k, |\varphi_k\rangle\}$ *that does not increase τ on average, i.e., for which*

$$\sum_k p_k \tau(|\varphi_k\rangle) \leq \tau(|\psi\rangle), \tag{3.19}$$

can be achieved by some TRIO.

Proof: Without loss of generality, we assume the states to be in the standard form,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(e^{i\theta/2}|0\rangle + e^{-i\theta/2}|1\rangle) \tag{3.20}$$

and

$$|\varphi_k\rangle = \frac{1}{\sqrt{2}}(e^{i\gamma_k/2}|0\rangle + e^{-i\gamma_k/2}|1\rangle), \tag{3.21}$$

with $\theta, \gamma_k \in [0, \pi/2]$.

We now define the state

$$|\bar{\varphi}\rangle \equiv \frac{1}{\sqrt{2}}(e^{i\gamma/2}|0\rangle + e^{-i\gamma/2}|1\rangle), \tag{3.22}$$

where γ is defined by

$$\cos\gamma \equiv \sum_k p_k \cos\gamma_k. \tag{3.23}$$

Noting that

$$\tau(|\varphi_k\rangle) = 1 - \cos\gamma_k \tag{3.24}$$

and

$$\tau(|\bar{\varphi}\rangle) = 1 - \cos\gamma, \tag{3.25}$$

we infer from Eq. (3.23) that

$$\tau(|\bar{\varphi}\rangle) = \sum_k p_k \tau(|\varphi_k\rangle). \tag{3.26}$$

From Eq. (3.19) we obtain $\tau(|\bar{\varphi}\rangle) \leq \tau(|\psi\rangle)$, which implies, by Theorem 6, that the transformation $|\psi\rangle \mapsto |\bar{\varphi}\rangle$ is achievable deterministically by TRIO. Therefore, we only need to show that we can generate the ensemble $\{(p_k, |\varphi_k\rangle)\}$ starting from $|\bar{\varphi}\rangle$.

For this purpose, we define a set of Kraus operators,

$$\left\{ K_k \equiv \begin{pmatrix} a_k & b_k \\ b_k & a_k \end{pmatrix} \right\}, \tag{3.27}$$

where

$$a_k = \frac{1}{2}\sqrt{p_k}\left[\frac{\cos(\gamma_k/2)}{\cos(\gamma/2)} + \frac{\sin(\gamma_k/2)}{\sin(\gamma/2)} \right] \tag{3.28}$$

and

$$b_k = \frac{1}{2}\sqrt{p_k}\left[\frac{\cos(\gamma_k/2)}{\cos(\gamma/2)} - \frac{\sin(\gamma_k/2)}{\sin(\gamma/2)} \right]. \tag{3.29}$$

Then it is straightforward to show that relation (3.23) implies that

$$\sum_k K_k^\dagger K_k = \mathbb{1}, \quad K_k|\bar{\varphi}\rangle = \sqrt{p_k}|\varphi_k\rangle. \tag{3.30}$$

Thus, the combination of this measurement with the deterministic protocol $|\psi\rangle \rightarrow |\bar{\varphi}\rangle$ realizes the required transformation $\mathcal{T}$.   □

Corollary 8: (Analog of Vidal's theorem[10]*) The maximum probability to convert $|\psi\rangle$ to $|\varphi\rangle$ by TRIO is given by*

$$P_{\max}(|\psi\rangle \mapsto |\varphi\rangle) = \min\left\{ \frac{\tau(|\psi\rangle)}{\tau(|\varphi\rangle)}, 1 \right\}. \tag{3.31}$$

Proof: Consider a transformation that takes $|\psi\rangle$ to $|\varphi\rangle$ with probability p and takes $|\psi\rangle$ to $|0\rangle$

with probability $1-p$. As $\tau(|0\rangle)=0$ (since it is an eigenstate of $\hat{\theta}$, albeit not in standard form), Theorem 7 implies that such a TRIO transformation exists if and only if

$$p \leq \frac{\tau(|\psi\rangle)}{\tau(|\varphi\rangle)}. \tag{3.32}$$

This corollary provides an operational interpretation of the measure τ.   □

IV. THE ASYMPTOTIC LIMIT

In Sec. III, the aim has been to determine whether a state is convertible to another state and what resources are required for this conversion. In this section we consider converting a state and its copies to another state plus copies.

Definition 4: A *copy* of a state $|\psi\rangle \in \mathcal{H}$ is a state $|\psi'\rangle$ in a different Hilbert space $\mathcal{H}'$, such that the description of $|\psi'\rangle$ is identical to the description of $|\psi\rangle$. The state with its copy is written $|\psi\rangle \otimes |\psi\rangle$, and n copies of the state are expressed as $|\psi\rangle^{\otimes n}$.

Specifically this section concerns transformations of the type

$$|\psi\rangle^{\otimes n} \to |\varphi\rangle^{\otimes m} \tag{4.1}$$

with the integers n and m allowed to tend to infinity. As discussed in Sec. II, the phase convention (2.8) respects composition of systems via the decomposition of tensor product states into total angular momenta. Here we will use such decompositions to show that arbitrary tensor powers of standard states can be brought to standard form with TRIO operations, i.e.,

$$|\psi\rangle^{\otimes n} = \left[\frac{1}{\sqrt{2}}(e^{i\theta/2}|0\rangle + e^{-i\theta/2}|1\rangle) \right]^{\otimes n} =: \frac{1}{\sqrt{2}}(e^{i\theta_n/2}|0\rangle + e^{-i\theta_n/2}|1\rangle). \tag{4.2}$$

We will give θ_n in terms of the original θ and finally define a frameness monotone that gives the maximum amount of distillable frameness from an ensemble.

It is not difficult to see that an arbitrary tensor product of standard states (4.2) can be written as a binomial distribution,

$$|\psi\rangle^{\otimes n} = \sum_{k=0}^{n} \sqrt{r_k}|\phi_k\rangle, \quad r_k = \frac{1}{2^n}\binom{n}{k}e^{i\theta(n-2k)}, \tag{4.3}$$

where $|\phi_k\rangle$ is the (normalized) equal superposition of all n-fold tensor products with k standard $|1\rangle$ states and $n-k$ standard $|0\rangle$ states,

$$|0\rangle := |\mu=1,\ell=0,m=0,\epsilon=+\rangle, |1\rangle := |\mu=1,\ell=1,m=0,\epsilon=+\rangle. \tag{4.4}$$

When decomposed, the states $|\phi_k\rangle$ will therefore contain total angular momenta up to $\ell=k$ with $m=0$.

From Eqs. (2.12) and (2.8), we can write such states as

$$|\mu\ell 0+\rangle = i^{\ell}|\mu\ell m\rangle. \tag{4.5}$$

Combining this with standard Clebsch–Gordan identities, we arrive at the following two decompositions that allow us to write any $|\phi_k\rangle$ in the preferred basis (2.5),

$$|\mu\ell 0+\rangle \otimes |0\rangle = |\mu'\ell 0+\rangle, \tag{4.6}$$

$$|\mu\ell 0+\rangle \otimes |1\rangle = \sqrt{\frac{\ell}{2\ell+1}}|\mu',\ell-1,0+\rangle + \sqrt{\frac{\ell+1}{2\ell+1}}|\mu'',\ell+1,0+\rangle, \qquad (4.7)$$

both are symmetric in the tensor factors, and normalization is of course preserved. All angular momenta resulting from such coupling are given their own multiplicity index because the resulting states must be orthogonal to any states previously occurring in a decomposition. Theorem 4 allows us to write the resulting state in the standard form (4.2).

Moreover, because

$$\sum_{k=0}^{n} r_k = \cos^n \theta \qquad (4.8)$$

is invariant under orthogonal transformations, by comparing the real norms of (4.2) and (4.3), we find that

$$\cos \theta_n = \cos^n \theta. \qquad (4.9)$$

This last observation motivates us to propose the following definition of an operational measure for the asymptotic TR resource.

Definition 5: Let $|\psi\rangle$ be a TR resource. We define

$$\tau^\infty(|\psi\rangle) \equiv -\log[1 - \tau(|\psi\rangle)] = -\log|\langle\psi^*|\psi\rangle| = -\log\cos\theta. \qquad (4.10)$$

Lemma 9: The measure τ^∞ is additive; that is,

$$\tau^\infty(|\psi\rangle|\phi\rangle) = \tau^\infty(|\psi\rangle) + \tau^\infty(|\phi\rangle). \qquad (4.11)$$

Proof:

$$\tau^\infty(|\psi\rangle|\phi\rangle) = -\log|\langle\psi^*|\psi\rangle\langle\phi^*|\phi\rangle| = \tau^\infty(|\psi\rangle) + \tau^\infty(|\phi\rangle). \qquad (4.12)$$

$\square$

Theorem 10: *Given two resources $|\psi\rangle$ and $|\phi\rangle$ and an integer n, let the integer m be the maximum integer such that the transformation $|\psi\rangle^{\otimes n} \rightarrow |\phi\rangle^{\otimes m}$ is possible by TRIO. Then,*

$$\lim_{n\to\infty} \frac{m}{n} = \frac{\tau^\infty(|\psi\rangle)}{\tau^\infty(|\phi\rangle)}. \qquad (4.13)$$

Proof: As τ is a frameness monotone and as τ^∞ is a monotonic function of τ, it follows that τ^∞ is also a monotone. Hence, the transformation $|\psi\rangle^{\otimes n} \rightarrow |\phi\rangle^{\otimes m}$ is possible by TRIO if and only if $\tau^\infty(|\psi\rangle^{\otimes n}) \geq \tau^\infty(|\phi\rangle^{\otimes m})$. This last inequality is equivalent to

$$\frac{m}{n} \leq \frac{\tau^\infty(|\psi\rangle)}{\tau^\infty(|\phi\rangle)}. \qquad (4.14)$$

Equality can obviously be achieved in the limit $n \to \infty$. $\square$

Note that even though τ^∞ is a monotone, it is *not* an ensemble monotone. This also happened with the chirality monotone in Ref. 4.

V. CONCLUSIONS

We have addressed the problem of breaking TR superselection by sharing a TR resource. Although TR invariance seems exotic because of its relevance to the counterintuitive problem of two parties trying to communicate but not knowing they are traveling the same or opposite directions in time, TR is pertinent to the more practical question of how to encode quantum information into states of motion when a shared motional reference is a consumable resource.

We solve the problem of finding quantum resources for breaking TR superselection by introducing TRIOs, which we call TRIOs, and thereby constructing TRIO equivalence classes. From these equivalence classes, we construct a resource theory for overcoming the superselection rule. The resource theory leads to three types of frameness monotones depending on whether the TRIO process should be deterministic, ensemble, or stochastic.

Whereas studies of reference frame resources have so far involved unitary representations, TR resources involve an antiunitary representation; thus our treatment broadens the scope of reference frame resource theories.

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