Chapter 2

Entanglement

2.1 The tensor product space

Consider two separate physical systems (we associate them with two fictitious observers, Alice and Bob) that are weakly interacting, or have been interacting in the past. In this case, they have to be treated as a single *bipartite* Hilbert space, which unites the Hilbert spaces associated with individual systems.

Suppose, for example, that Alice has[1] a horizontally polarized photon, $|H\rangle$, while Bob has a photon in state $|V\rangle$. Then we say that the joint state of Alice's and Bob's photon is

$$|H\rangle \otimes |V\rangle \equiv |H_A V_B\rangle \equiv |H\rangle\,|V\rangle \equiv |H\,V\rangle\,. \tag{2.1}$$

These joint states are called the *tensor product* states.

The Hilbert space that encompasses tensor product states is, however, not restricted by these states. For example, since it contains states $|HV\rangle$ and $|VH\rangle$ and it is a linear space, it must also contain state $(|HV\rangle - |VH\rangle)/\sqrt{2}$. This is a physical state, because it has norm 1. But it can no longer be interpreted as a tensor product state, i.e. a combination of Alice's photon being in one state and Bob's in another. This is a *nonlocal superposition*, or *entangled* state: a quantum superposition of two situations: a situation where Alice has a horizontal photon while Bob has a vertical one, and vice versa. If they measure their photons' polarizations in the canonical basis, they will *always* detect different polarizations.

This correlation by itself is not too amazing. For example, even in the classical world we can play a game in which we give Alice and Bob a pair of marbles, one of which is red and the other white. The marbles are packed in opaque boxes, so their color cannot be seen. Then Alice flies to Venus and Bob flies to Mars, where they open their boxes. As soon as they do, each of them will instantly learn the color of their counterpart's marble, even though they are millions of miles apart.

But properties of quantum superpositions extend beyond this simple picture. In addition to polarization correlations, there is a certain phase relation, signified by the negative sign between terms $|HV\rangle$ and $|VH\rangle$. In this way, state $(|HV\rangle + |VH\rangle)/\sqrt{2}$ is distinctly different from, e.g. $(|HV\rangle - |VH\rangle)/\sqrt{2}$, even though both exhibit similar correlations when measured in the canonical basis. And it is this phase relation that gives rise to a whole range of classically unthinkable *nonlocal quantum phenomena*, which are the main subject of this chapter.

Before we study these phenomena, we need to sharpen our pencils and upgrade our theoretical machinery so it can be applied to tensor product Hilbert spaces. Although we will do our derivations for bipartite tensor products, they can be straightforwardly extended to multipartite systems involving three and more Hilbert spaces.

[1] This is a metaphoric statement, of course. Photons move at a speed of light, and no one can "have" them for any extended time period. The notion of Alice or Bob "having" a photon corresponds, as a rule, to an instant in time just before the measurement.

Definition 2.1 *Tensor product* $\mathbb{V}\otimes\mathbb{W}$ of Hilbert spaces $\mathbb{V}$ and $\mathbb{W}$ is a Hilbert space consisting of elements $|a\rangle\otimes|b\rangle$ (with $|a\rangle\in\mathbb{V}$ and $|b\rangle\in\mathbb{W}$) and their linear combinations. The operations in the space obey the following rules:

1. $\lambda(|a\rangle\otimes|b\rangle)=(\lambda|a\rangle)\otimes|b\rangle=|a\rangle\otimes(\lambda|b\rangle)$;
2. $(|a_1\rangle+|a_2\rangle)\otimes|b\rangle=|a_1\rangle\otimes|b\rangle+|a_2\rangle\otimes|b\rangle$;
   $|a\rangle\otimes(|b_1\rangle+|b_2\rangle)=|a\rangle\otimes|b_1\rangle+|a\rangle\otimes|b_2\rangle$;
3. The inner product of two states $|a\rangle\otimes|b\rangle$ and $|a'\rangle\otimes|b'\rangle$ in $\mathbb{V}\otimes\mathbb{W}$ is given by $\langle ab|\,a'b'\rangle=\langle a|\,a'\rangle\langle a'|\,b'\rangle$.

Exercise 2.1 Show that $\forall|a\rangle\in\mathbb{V},\ \forall|b\rangle\in\mathbb{W}$,
$|\text{zero}\rangle_{\mathbb{V}}\otimes|b\rangle_{\mathbb{W}}=|a\rangle_{\mathbb{V}}\otimes|\text{zero}\rangle_{\mathbb{W}}=|\text{zero}\rangle_{\mathbb{V}\otimes\mathbb{W}}$.

Exercise 2.2 Given the bases $\{|v_i\rangle\}_{i=1}^N$ and $\{|w_i\rangle\}_{i=1}^M$ in $\mathbb{V}$ and $\mathbb{W}$, respectively, construct an orthonormal basis in $\mathbb{V}\otimes\mathbb{W}$. What is the dimension of $\mathbb{V}\otimes\mathbb{W}$?
Answer: The basis is a set of tensor products $\{|v_i\rangle\otimes|w_j\rangle\}$. The dimension of the tensor product space is the product NM of its components' dimensions.

The polarization Hilbert space of two photons is four-dimensional. The canonical orthonormal basis of this space is $\{|HH\rangle,|HV\rangle,|VH\rangle,|VV\rangle\}$. For example, when Alice has a photon with a $+45°$ polarization, and Bob with a right circular polarization, the state is $|+\rangle\otimes|R\rangle$, which can be decomposed into the canononical basis according to $|+R\rangle=(|H\rangle+|V\rangle)/\sqrt{2}\otimes(|H\rangle+i|V\rangle)/\sqrt{2}=(|HH\rangle+i|HV\rangle+|VH\rangle+i|VV\rangle)/2$.

Sets $\{|++\rangle,|+-\rangle,|-+\rangle,|--\rangle\}$ as well as $\{|RR\rangle,|RL\rangle,|LR\rangle,|RR\rangle\}$ are also examples of orthonormal bases in this Hilbert space.

Exercise 2.3 a) Find the inner product $\langle\Pi|\,\Omega\rangle$, where $|\Pi\rangle=5|HH\rangle+6i|HV\rangle+7i|VV\rangle$ and $|\Omega\rangle=2|HH\rangle+3|HV\rangle+4|VH\rangle$.

b) Find the inner product $\langle\Pi|\,\Omega\rangle$, where $|\Pi\rangle=(2|H\rangle+i|V\rangle)\otimes(i|H\rangle-|V\rangle)$ $|\Omega\rangle=(2i|H\rangle-3i|V\rangle)\otimes(|H\rangle+|V\rangle)$.

Exercise 2.4 Do the sets

a) $\{|H-\rangle,|H+\rangle,|V-\rangle,|V+\rangle\}$,

b) $\{|H-\rangle,|H+\rangle,|VR\rangle,|VL\rangle\}$, and

c) $\{|H-\rangle,|HH\rangle,|VR\rangle,|VL\rangle\}$

form bases in the two-photon Hilbert space? Are they orthonormal?

Definition 2.2 Elements of $\mathbb{V}\otimes\mathbb{W}$ that can be presented in the form of a tensor product $|a\rangle\otimes|b\rangle$ are called *separable*. Others are called *entangled*.

Note 2.1 According to the definition of the tensor product, any entangled vector can be expressed as a linear combination of separable vectors.

Exercise 2.5 Write the matrix representation in the canonical basis of the state in which Alice has a 30° polarized photon, and Bob a right circularly polarized photon. Is this state separable or entangled?

Exercise 2.6 Show that the states

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|HV\rangle+|VH\rangle) \quad (2.2)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle-|VH\rangle) \quad (2.3)$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle+|VV\rangle) \quad (2.4)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|HH\rangle-|VV\rangle) \quad (2.5)$$

are entangled.

Definition 2.3 The four states (2.2)–(2.5) are called the *Bell states.*

Exercise 2.7 Show that the four Bell states form an orthonormal basis.

Exercise 2.8 Can Alice and Bob distinguish the symmetric Bell states $|\Phi^-\rangle$ and $|\Phi^+\rangle$ by measuring them in the canonical basis? What about the diagonal basis $\{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle\}$? Prove your answer by decomposing these Bell states into the diagonal basis.

Definition 2.4 *Tensor product of operators* $\hat{A}$ in $\mathbb{V}$ and $\hat{B}$ in $\mathbb{W}$ is a linear operator $\hat{A}\otimes\hat{B}$ in $\mathbb{V}\otimes\mathbb{W}$ such that $\forall\,|\Psi\rangle = \sum_i \lambda_i\,|a_i\rangle\otimes|b_i\rangle \in \mathbb{V}\otimes\mathbb{W},\ (\hat{A}\otimes\hat{B})\,|\Psi\rangle \equiv \sum_i \lambda_i(\hat{A}\,|a_i\rangle)\otimes(\hat{B}\,|b_i\rangle)$.

In other words, component $\hat{A}$ of the tensor product operator acts on Alice's Hilbert space while component $\hat{B}$ on Bob's.

Tensor product operators of the form $\hat{A}\otimes\hat{\mathbf{1}}$ or $\hat{\mathbf{1}}\otimes\hat{B}$ are called *local operators* because they affect only one of the component Hilbert spaces. An example is a waveplate placed into the path of Alice's photon, so its polarization is turned while that of Bob's photon is not. Sometimes a simplified notation is used: one writes just $\hat{A}$ instead of $\hat{A}\otimes\hat{\mathbf{1}}$ and $\hat{B}$ instead of $\hat{\mathbf{1}}\otimes\hat{B}$.

For matrices of states and operators in tensor-product spaces, the double-index notation is often used. If $\{|v_i\rangle\}$ and $\{|w_j\rangle\}$ are bases in Alice's and Bob's Hilbert spaces, respectively, then, as we know, $\{|v_i\,w_j\rangle\}$ is a basis in the tensor product space. Rather than defining a single index enumerating the elements of the new basis, we use double indices to identify the matrix elements:

$$(A\otimes B)_{iji'j'} = \left\langle v_i w_j \middle| \hat{A}\hat{B} \middle| v_i' w_j' \right\rangle. \tag{2.6}$$

This is an extension of the rule (1.26) for single-party Hilbert spaces.

Exercise 2.9 Express the matrix of the tensor product operator $\hat{A}\otimes\hat{B}$ in the basis $\{|v_i\rangle\otimes|w_j\rangle\}$ through the matrices of operators $\hat{A}$ and $\hat{B}$ in the respective bases $\{|v_i\rangle\}$ and $\{|w_j\rangle\}$.
Answer: For each matrix element,

$$(\hat{A}\otimes\hat{B})_{iji'j'} = A_{ii'}B_{jj'}. \tag{2.7}$$

Exercise 2.10 Show that, for operators $\hat{A}_1, \hat{A}_2$ in $\mathbb{V}$, $\hat{B}_1, \hat{B}_2$ in $\mathbb{W}$

$$\hat{A}_1\hat{A}_2\otimes\hat{B}_1\hat{B}_2 = (\hat{A}_1\otimes\hat{B}_1)(\hat{A}_2\otimes\hat{B}_2)$$

Definition 2.5 We define the *adjoint tensor product space* analogously to the direct tensor product space, i.e. $\forall\,|ab\rangle$,

$$\text{Adjoint}(|a\rangle\otimes|b\rangle) \equiv \text{Adjoint}(|a\rangle)\otimes\text{Adjoint}(|b\rangle) \equiv \langle a|\otimes\langle b| \equiv \langle ab|\,. \tag{2.8}$$

Exercise 2.11 Show that, for $\hat{A}$ in $\mathbb{V}$, $\hat{B}$ in $\mathbb{W}$: $(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger$

Exercise 2.12 Show that:

a) tensor product of two Hermitian operators is Hermitian;

b) tensor product of two unitary operators is unitary.

Exercise 2.13 For $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$,

a) Find the expectation value and the uncertainty of the operator $\hat{\sigma}_{x_A}\otimes\hat{\sigma}_{y_B}$;

b) Find the probability of detecting the state $|R\rangle_A\,|{-30^\circ}\rangle_B$;

c) Find the probability of detecting the state $\frac{1}{3}(|HV\rangle + 2\,|VH\rangle + 2\,|VV\rangle)$.

(For the last two parts, it is assumed that the measurement is performed in some orthonormal basis that contains the state we are interested in.)

2.2 Quantum computation

The idea of quantum computation is to use so-called *quantum bits (qubits)* as basic units of information. A qubit is any physical system whose Hilbert space is described by a 2-dimensional Hilbert space — for example, polarization of the photon. In contrast to a classical bit, a qubit can be not only in a definite state $|0\rangle$ or $|1\rangle$, but also in a superposition of these states. Accordingly, multiple qubits can be in entangled nonlocal superposition states.

It is the entanglement that makes the quantum computer much more powerful than a classical one. Consider, for example, three qubits in a superposition

$$a_{000}\,|000\rangle + a_{001}\,|001\rangle + a_{010}\,|010\rangle + a_{011}\,|011\rangle + a_{100}\,|100\rangle + a_{101}\,|101\rangle + a_{110}\,|110\rangle + a_{111}\,|111\rangle\,. \quad (2.9)$$

Performing a set of logical operations with these three qubits in this state, we simultaneously perform them with all $2^3 = 8$ sets of qubit values contained in state (2.9). In this way, we achieve an exponential degree of parallelism in our calculations. For example, even a tiny, 30-qubit quantum computer will perform a billion ($2^{10} \approx 10^9$) times faster than its classical counterpart.

Of course, quantum computation is not as simple as it may appear from this example. Problems arise both on theoretical and practical fronts. Just as an example of the many fundamental issues, let me mention the following problem. Suppose the quantum computer has completed its calculation, generating a superposition of the answers associated with all the inputs. But if we now try to measure this state, the only thing we will obtain is a random element of the measurement basis. That is, even though we may have a result of the calculation stored in our state, we are unable to read it out!

Thus it turns out that only a very limited (but important) class of problems can be efficiently solved on a quantum computer. Furthermore, the quantum computer is very difficult to build. As we discussed in Section 1.8, and discuss in further detail below, any interaction of a quantum state with the environment may constitute an "inadvertent" measurement, which will collapse the state and take us back to square one. The likelihood of this to happen is especially high for a multipartite entangled state, because interaction of *any* of the qubits with the environment will affect the entire superposition (more on this in the next section).

This is one of the primary reasons why quantum computation technology has been developing so slowly. At the moment, we don't even know what physical system is best suitable as the carrier of quantum information! Different research groups around the world are investigating different systems — trapped atoms and ions, superconducting junctions, quantum dots and even liquids — to uncover their potential for this role. As it turns out, the photon also turns out a promising candidate. This is because the average energy of the optical photon (2–4 electronvolts) corresponds to a few tens of thousand Kelvins, i.e. much higher than the typical temperature of our environment. As a result, photons are not too likely to interact with this environment, thereby losing the quantum information they are carrying. And it is easy to *encode* the qubit in the polarization of a photon: for example, logical state $|0\rangle$ can correspond to the horizontal polarization and state $|1\rangle$ to vertical.

It is also easy to implement single-qubit logical operations with this encoding. For example, we can perform the logical "not" operation using a $\lambda/2$ plate with its optical axis oriented at angle 45° to horizontal: state $|0\rangle$ ($|H\rangle$) will become $|1\rangle$ ($|V\rangle$) and vice versa (see Ex. 1.48). However, in order to allow a full range of computations accessible to a classical computer (Turing machine), we also need operations in which qubits could interact: the state of one qubit would affect the state of another. Theoretical quantum computing research has shown that in order to build a full quantum computer, it is sufficient, in addition to single-qubit operations, to be able to implement only one two-qubit operation: the *conditional "not" gate*, or the *c-not gate.*

The c-not gate involves two qubits: *control* and *target.* If the state of the control qubit is $|0\rangle$, the gate does not change the qubit values. But if the control qubit is $|1\rangle$, the value of the target "flips": $|0\rangle$ becomes $|1\rangle$ and $|1\rangle$ becomes $|0\rangle$. This corresponds to the following truth table:

Note that the output value of the target qubit corresponds to the result of the classical exclusive-or gate.

In order to figure out how to build the c-not gate, let us solve the following two Exercises.

Table 2.1: The truth table of the c-not gate.

input		output	
control	target	control	target
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Exercise 2.14 Write the operators corresponding to the following operations on a pair of photons. Logical state $|0\rangle$ is encoded by the horizontal polarization and state logical state $|1\rangle$ by vertical.

a) The c-not gate.

b) An operation that leaves states $|00\rangle$, $|01\rangle$, $|10\rangle$ unchanged, but multiplies state $|11\rangle$ by a phase factor of -1 (*the conditional phase shift, or c-phase gate*).

c) Action of a $\lambda/2$ plate with its optical axis at angle $\pi/8$ to horizontal (i.e. the Hadamard gate, see Ex. 1.54) *only* upon the second (target) photon.

Are these operators unitary?

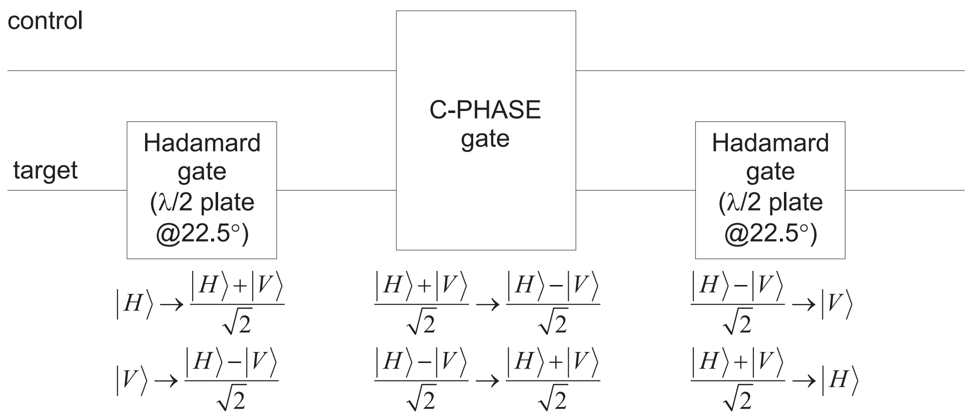

Figure 2.1: Implementation of the c-not gate using a c-phase gate and two Hadamard gates (Ex. 2.15). The transformations underneath the diagram are for the case the control qubit is in state $|V\rangle$, i.e. logical $|1\rangle$.

Exercise 2.15 Show that the c-not gate can be constructed by applying, in sequence, a Hadamard gate in Bob's space, a conditional phase shift, and a Hadamard gate in Bob's space again.

Through these results, the task of constructing the c-not gate becomes somewhat simpler. Rather than having to flip the qubits, we just need to change their phases. But even that is extremely difficult. In order to implement the c-phase gate, we need a medium in which a photon would "sense" different indices of refraction dependent on the polarization of another photon present in that medium. This is not what we would normally observe in optics: typically, if several light waves are present in the same medium, they will not interact, but propagate independently of the other waves' presence. Although mutual influence of electromagnetic waves upon each other has been observed (this class of phenomena is studied by *nonlinear optics*), in usual media, such as glass or crystals, it is present only if at least one of the fields is extremely strong, on the scale of trillions of photons. Making nonlinear optical effects significant at a few photons' optical intensity level is a difficult problem and is currently investigated by many research groups.

Exercise 2.16 Show that

a) a local operator, or a tensor product of local operators cannot make an entangled state out of a separable one, and vice versa.

b) the operators from Ex. 2.14 (a,b) can create an entangled state out of a separable one.

2.3 Local measurements of entangled states

Quantum measurements in bipartite systems proceed according to the Second postulate, if both observers perform measurements on their respective Hilbert spaces. However, since the observers of the bipartite system are independent, it is possible that only one of them (e.g. Alice) performs the measurement while the other one (Bob) does not. We call this a *local measurement.*

Studying local measurements that are performed on entangled states can lead us to many interesting conclusions. Let us consider a simple example. Suppose Alice measures state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$ in the canonical basis. Because $|\Psi^-\rangle$ contains, with equal amplitudes, states $|HV\rangle$ and $|VH\rangle$, Alice will be equally likely ($\mathrm{pr}_H = \mathrm{pr}_V = 1/2$) to observe either horizontal or vertical polarization. If she observes a horizontally polarized photon, we can state with certainty that Bob's photon is vertically polarized, so its state becomes $|V\rangle$, and vice versa.

If we think of this experiment in classical terms, we won't discover anything strange (remember our example with marbles in Sec. 2.1). But if we look at the philosophical side of its quantum description, a little red flag emerges. Indeed, let us think of Bob's photon as a physical object. Before Alice's measurement, it is a part of an entangled entity, so one could not associate any particular quantum state with this photon alone. But after the measurement, Alice's portion of the state collapses onto a specific element of her measurement basis, so the state stops being entangled. This means that Bob's photon will also change, collapsing into a well-defined quantum state. So it appears that Alice, by doing something on Venus, can change (at least mathematically) the state of affairs on Mars.

Is this just a mathematical hiccough or is there any deep physics involved? In order to answer this question, we need to rigorously define the notion of local measurement. To this end, we recall the alternative formulation of the Second Postulate of quantum physics in terms of projection operators (see Definition 1.24). A measurement of a (single-party) state $|\psi\rangle$ in basis $\{|v_i\rangle\}$ is equivalent to the action of one of the projection operators on that state:

$$|\psi_{out}\rangle = \hat{P}_i\,|\psi\rangle \tag{2.10}$$

with the probability that is equal to the square of the norm of the resulting state: $\mathrm{pr}_i = \langle\psi_{out}|\,\psi_{out}\rangle = \langle\psi|\,v_i\rangle\,\langle v_i|\,\psi\rangle$.

Local measurements of bipartite states are described in the same terms, except that the projection operator is applied locally, in the sense of Definition 2.4. More specifically, Alice's local measurement on a bipartite state $|\Psi\rangle$ in a basis $\{|v_i\rangle\}$ will collapse $|\Psi\rangle$ onto

$$|\Psi_{i,out}\rangle = \left(\hat{P}_i \otimes \hat{\mathbf{1}}\right)|\Psi\rangle = |v_i\rangle_{\text{Alice}} \otimes \langle v_i|\,\Psi\rangle_{\text{Bob}} \tag{2.11}$$

(where i is random) with a probability

$$\mathrm{pr}_i = \langle\Psi_{i,out}|\,\Psi_{i,out}\rangle = \langle\Psi|\,v_i\rangle\,\langle v_i|\,\Psi\rangle\,. \tag{2.12}$$

We see that, upon a local measurement, an entangled bipartite state will collapse into a separable one. If Alice destroys her system in the process of the measurement, the resulting quantum state, $\langle v_i|\,\Psi\rangle$, will be localized with Bob.

The state $|\Psi_{i,out}\rangle$ that is produced by the measurement is not normalized to 1. This reflects the fact that each local measurement result and the corresponding quantum state will emerge with non-unity probability pr_i. One should be careful with this interpretation, though: after the measurement has taken place and the result is *known*, the state $|\Psi_{out}\rangle$ exists with certainty and should be renormalized to a unity norm.

One additional clarification that is required before we can employ the above extension of the Second postulate to local measurements is a rigorous definition of the inner product $\langle v_i|\ \Psi\rangle$ between a vector in Alice's Hilbert space and a vector in the tensor product space, such as in Eq. (2.11). This definition is rather intuitive.

Definition 2.6 The *partial inner product* between a local state $|a\rangle$ in Hilbert space $\mathbb{V}$ and a bipartite state $|\Psi\rangle = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle$ in Hilbert space $\mathbb{V} \otimes \mathbb{W}$ is a state in Hilbert space $\mathbb{W}$ given by

$$\langle a|\ \Psi\rangle \equiv \sum_i \lambda_i \langle a|\ a_i\rangle |b_i\rangle ; \tag{2.13a}$$

$$\langle \Psi|\ a\rangle \equiv \sum_i \lambda_i^* \langle a_i|\ a\rangle \langle b_i| . \tag{2.13b}$$

This definition is analogous for the partial inner product of $|\Psi\rangle$ with a local state in space $\mathbb{W}$.

Exercise 2.17 For $|\psi\rangle = 2|H\rangle + i|V\rangle$, find $\langle \psi_B|\ \Omega\rangle$ and $\langle \Pi|\ \psi_A\rangle$, where $|\Omega\rangle = 2|HH\rangle + 3|HV\rangle + 4|VH\rangle$, $|\Pi\rangle = (2|H\rangle + i|V\rangle) \otimes (i|H\rangle - |V\rangle)$ and $|\Omega\rangle = (2i|H\rangle - 3i|V\rangle) \otimes (|H\rangle + |V\rangle)$ and subscripts A and B mean that state $|\psi\rangle$ is localized in Alice's or Bob's space, respectively.

Exercise 2.18 Suppose $|\Psi\rangle$ is a state in the tensor product space, $|a\rangle$ and $|b\rangle$ are states in Alice's and Bob's spaces, respectively. Show that

$$\langle a| (\langle b|\ \Psi\rangle) = \langle b| (\langle a|\ \Psi\rangle) = \langle ab|\ \Psi\rangle \tag{2.14}$$

Exercise 2.19 For each Bell state, show that a local measurement by Alice in *any* orthonormal basis will yield either result with a probability of 1/2.

Exercise 2.20 Let Alice and Bob share a bipartite state $|\Psi\rangle$. Both Alice and Bob perform measurements on their portion of $|\Psi\rangle$, in bases $\{|v_i\rangle\}$ and $\{|w_j\rangle\}$, respectively. These measurements can occur according to three scenarios:

a) Alice and Bob perform their measurements simultaneously, so the original Second Postulate applies for a projective measurement of the state $|\Psi\rangle$ in the basis $\{|v_i\rangle \otimes |w_j\rangle\}$

b) Alice performs her measurement first, so Eq. (2.11) applies, and then Bob measures the resulting state.

c) Bob performs her measurement first, and then Alice measures the resulting state.

Show that the set of probabilities pr_{ij} (i.e. the probabilities that Alice detects $|v_i\rangle$ while Bob detects $|w_j\rangle$) is the same for each of these scenarios.

Exercise 2.21 Suppose Alice and Bob share state $|\Psi^-\rangle$. Alice measures her portion of the state in the basis $\{|\theta\rangle, \left|\frac{\pi}{2}\right\rangle + \theta\}$. Show that

a) if Alice detects $|\theta\rangle$, Bob's state becomes $|\pi/2 + \theta\rangle$;

b) if Alice detects $|\pi/2 + \theta\rangle$, Bob's state becomes $|\theta\rangle$;

c) each of these results happens with probability 1/2.

The above result is remarkable. By choosing the tilt angle θ of the measurement basis, Alice can *remotely prepare* an arbitrary linear polarization state (with a $\pm 90°$ ambiguity) at Bob's location. This happens in spite of the fact that Alice and Bob can be millions of miles away from each other, and have no opportunity to interact. Furthermore, this happens instantly, i.e. faster than the speed of light!

This appears to be in outrageous contradiction not only with all the physics we have previously known, but with the simple common sense. How can it be possible to instantly change something that is located far away, without any possibility to interact with the that location?

Perhaps the very first question a diligent physics student would ask is whether this conclusion has been verified experimentally. The answer is affirmative. In the last quarter of the twentieth century, physicists in many parts of the world studied different versions of the remote state preparation effect. Some of the setups were arranged so that Alice's and Bob's laboratories were located hundreds of meters away from each other, and their measurements were ensured to occur within a spacelike interval, in order to exclude even theoretical possibility for Alice to affect Bob's state through any interaction known in nature. All these experiments unequivocally confirmed the validity of the quantum mechanics' predictions.

Let us ask a more practical question, though. Can Alice make any physically observable change to Bob's system or transfer any information to Bob in this fashion? At the first sight, this seems to be the case. Indeed, Alice can choose any arbitrary angle θ for her measurement basis. The set of all possible values of this angle is a continuum, and thus a precise value of θ, which is transferred to Bob in the form of the polarization angle of the photon remotely prepared at his location, contains infinite information (even if we take into account the $\pm 90^\circ$ ambiguity — since Bob does know Alice's result). But can Bob extract this information?

By measuring the polarization of his photon, in any basis, Bob can extract just one bit of that information — dependent on which of his single-photon detectors clicks. This is a fundamental restriction, and it is valid independently of how precisely Alice has measured θ. Furthermore, the $\pm 90^\circ$ ambiguity associated with the remote state preparation makes even this single bit useless. To see this, let us solve the following problem.

Exercise 2.22 In the setting of Ex. 2.21, Bob measures the polarization of his photon in the canonical basis immediately after Alice's measurement. What is the probability of each result given that Bob does not know the result of Alice's measurement?
Answer: $\mathrm{pr}_{\mathrm{Bob},H} = \mathrm{pr}_{\mathrm{Bob},V} = 1/2$.

This answer means that, without knowing the outcome of Alice's measurement, Bob can extract *no information whatsoever* about Alice's actions. Although instant remote state preparation is predicted by theory and confirmed by experiment, it cannot be used for superluminal, interaction-free communication.

But how can we verify the effect of remote state preparation experimentally if there is no information transfer? The answer is that we can repeat the experiment many times and perform quantum state tomography on Bob's photon (see Ex. 1.33), taking into account only those instances in which Alice detected, say, $|\theta\rangle$. In this case Bob will reconstruct state $|\pi/2+\theta\rangle$ with arbitrarily high precision. For this procedure, however, Bob will need classical input from Alice regarding which of her measurements resulted in detecting $|\theta\rangle$, so the information transfer is neither instantaneous nor interaction-free.

Interestingly, this experiment will give the same result even if Bob performs his measurements *before* Alice does. If he still takes into account (*postselects*) only those events in which Alice observed $|\theta\rangle$, he will still reconstruct state $|\pi/2+\theta\rangle$, even though this state has been remotely prepared by Alice *after* it has been measured by Bob. This follows from Ex. 2.20: no matter in which order Alice and Bob perform their measurements, the probability for each pair of results does not change. And since it is the probabilities that determine the measurement statistics (i.e. how often each result occurs), and it is the statistics that we use for tomography, the reconstructed state will be the same. This extremely counterintuitive conclusion is perhaps easier to come to terms with if we remember that, according to the special theory of relativity, the time order of a pair of events separated by a spacelike interval depends on the frame of reference chosen. by choosing the frame we can decide which event occurs first.

It thus appears that the effect of instant remote state preparation in each particular instance has no practical circumstances: Alice's measurement does not change any measurable physical properties of Bob's photon. Quantum mechanics *misleads* us into thinking otherwise, by telling us that Bob's *state* after Alice's measurement depends on the setup of that measurement. But in fact, the quantum state is a pure theoretical construct, and it is never directly observed in an experiment. We can infer information about the state only through *indirect* inference from the statistic obtained in multiple measurements.

So perhaps we can dismiss the concept of the quantum state altogether and invent another theory, which explains experimental results equally well but does not involve theoretical concepts that contradict the common sense? We give the answer to this question in the next section.

Exercise 2.23 Suppose Alice measures $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$ in the circular polarization basis. Upon which state will Bob's photon project for each of Alice's results?

Exercise 2.24 Let $|\theta\rangle$ be the linear polarization state at angle θ to horizontal. Show that for any θ, the state $|\Psi^-\rangle = |H_A V_B\rangle - |V_A H_B\rangle$ can be expressed as $|\Psi^-\rangle = \left|\theta_A(\frac{\pi}{2}+\theta)_B\right\rangle - \left|(\frac{\pi}{2}+\theta)_A\theta_B\right\rangle$.

Note 2.2 This shows that the state $|\Psi^-\rangle$ is *isotropic*, i.e. it remains the same, no matter which direction we define as horizontal (as long as it is perpendicular to the direction of the photons' propagation, of course). This property of $|\Psi^-\rangle$ is unique among all the Bell states.

Exercise 2.25 Suppose Alice and Bob share state $|\Psi^-\rangle$. Alice wishes to remotely prepare some linear superposition $\alpha|H\rangle + \beta|V\rangle$ at Bob's station, where α and β are arbitrary but $|\alpha|^2 + |\beta|^2 = 1$ (i.e. the output state is normalized). In which basis should she measure? What is the probability of success?

Let us now consider a situation that takes place when Alice loses her share of the entangled state or fails to provide us with her measurement results. The photon is absorbed on its way towards Alice's detector, or the detector fails to function, or Alice simply lets her photon fly out of the lab window into the sky, where in may eventually get measured by distant aliens. What can we say about the quantum state of Bob's photon?

One thing we do know (Ex. 2.20) is that no matter what happened to Alice's photon, the experimentally measurable properties of Bob's photon do not change. Therefore, as long as we are interested in describing Bob's photon, we can make any convenient assumption about the fate of Alice's photon. For example, let us assume that Alice has measured it in the canonical basis and did not tell us the result.

Specializing, again, to the initial state being $|\Psi^-\rangle$, we know that, Alice could, with probability 1/2, detect either $|H\rangle$ (in which case Bob's photon projects onto $|V\rangle$) or $|V\rangle$ (in which case Bob's photon projects onto $|H\rangle$). But since we do not know Alice's result, we can only describe the state of Bob's photon verbally as "either $|H\rangle$ with probability 1/2 or $|H\rangle$ with probability 1/2".

This is about the best we can do. Assuming other bases that Alice could have used, we could also describe Bob's photon as "either $|+45^\circ\rangle$ with probability 1/2 or $|-45^\circ\rangle$ with probability 1/2" or "either $|R\rangle$ with probability 1/2 or $|L\rangle$ with probability 1/2" and so on. All these descriptions are equivalent. The polarization of Bob's photon is *completely mixed* — similar to that of natural light. There is no single state vector describing this polarization.

In Chapter 5 we will study properties of mixed states and ways to describe them mathematically. It is important to understand now, though, that if we lose a part of an entangled state, the remaining part stops being in a superposition state and becomes just a statistical mixture. It has now lost its quantum superposition properties and its description can now be given in terms of the classical theory of probabilities rather than quantum mechanics. This phenomenon is known as *decoherence* and is particularly relevant to quantum computation. This is because quantum computers contain large, complex entangled states and if even one of them is lost (or have entangled itself with the environment), the whole superposition will collapse.

Exercise 2.26 Alice and Bob share an entangled two-photon state:

a) $|\Psi\rangle = (|HH\rangle + 2|VV\rangle)/\sqrt{5}$;

b) $|\Psi\rangle = (|HH\rangle + |HV\rangle + |VV\rangle)/\sqrt{3}$;

Suppose Alice's photon has been lost. Give a verbal description of the state of Bob's photon.

2.4 An insight into quantum measurements

Let us examine the Second Postulate of quantum mechanics more closely. It has a strange feature: although quantum physics is supposed to be more general than classical and include the latter as a special case, we need to resort to classical terms in order to explain quantum mechanics[2]. Indeed, the postulate says a measurement converts a quantum superposition into a classical probabilistic mixture. But how can this be the case if *everything* is quantum?

In order to answer this question and try reformulating the Second Postulate in quantum terms, let us recall the Schrödinger cat paradox in its original form. In fact, it is something more complex than just a superposition of the dead and alive states of a cat. Here is a quote from the 1935 Schrödinger's article in the German magazine Naturwissenschaften ("Natural Sciences").

A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter, there is a tiny bit of radioactive substance, so small that perhaps in the course of the hour, one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges, and through a relay releases a hammer that shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The psi-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts.

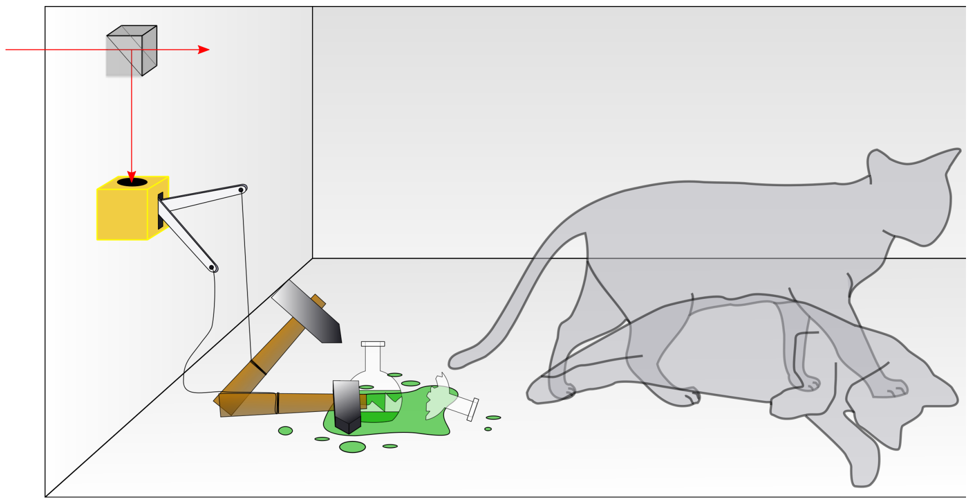

Figure 2.2: The Schrödinger cat Gedankenexperiment (adapted from *Wikipedia*).

Figure 2.2 shows the situation envisioned by Schrödinger with a minor amendment: instead of a radioactive atom we have a single photon in the superposition state of horizontal and vertical polarizations, for example,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle). \tag{2.15}$$

Let us restate Schrödinger's example in the modern language. The part of the photon corresponding to the vertical polarization term reflects off the PBS and hits the sensitive area of the photodetector, triggering an avalanche of photoelectrons. This means that the photon *entangles* itself with the photodetector, bringing about the state (for the sake of the argument let us assume that the photon is not destroyed at the time of detection)

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|H\rangle \otimes |\text{no avalanche}\rangle + |V\rangle \otimes |\text{avalanche}\rangle). \tag{2.16}$$

[2]This standard approach to quantum measurements is referred to as *Copenhagen interpretation* in honor of Niels Bohr.

Subsequently, the ampule with the poison, and, finally, the cat also become a part of that entangled state:

$$\begin{aligned} |\Psi\rangle \quad = \quad & \frac{1}{\sqrt{2}}(|H\rangle \otimes |\text{no avalanche}\rangle \otimes |\text{flask intact}\rangle \otimes |\text{cat alive}\rangle \\ & + |V\rangle \otimes |\text{avalanche}\rangle \otimes |\text{flask shattered}\rangle \otimes |\text{cat dead}\rangle). \end{aligned} \tag{2.17}$$

If this experiment is thoroughly isolated from the outside world, the propagation of entanglement stops here.

Let us look at state (2.17) from two points of view. For the cat inside the box, the process described by Schrödinger constitutes a measurement of the photon's state, potentially with fatal consequences. The cat *does not know* it is in a superposition state: the "part" of the cat that is alive sees only an intact flask, a detector that has not clicked and a horizontally polarized photon. It does not know, and has no possibility to know, that there exists another part of the superposition which is dead, because everything it can observe (the photon, the detector and the flask) are consistent with it being alive. Similarly, if the dead cat could make observation, it would see all the objects in the state consistent with it being dead, and no opportunity to see the "alive" part of the superposition.

For the experimentalist outside the box, the situation is different. This experimentalist could, in principle, see both parts of the superposition and verify that the quantum state inside the box is indeed a superposition. To do so, (s)he would have to apply a Hamiltonian that reverses the evolution and brings state (2.17) back to state (2.15) — and then measure the photon in the diagonal basis. Such an experiment is of course extremely difficult, far beyond our technical capability. But theoretically it is possible.

We thus see that the quantum measurement in the sense of the Second Postulate is an evolution which entangles the measured object with the measurement apparatus and eventually with the experimentalist. Once the experimentalist enters each term of this entangled superposition, his or her "copy" within each term stops seeing other terms. At that moment, in view of that experimentalist, the superposition collapses.

Measuring state (2.17) is difficult not only because one would need to inverse the evolution. In writing Eq. (2.17) we assumed that the box, in which the experiment with the cat is enclosed, is perfectly isolated from the outside world. This is, of course, not realistic. A living cat generates heat, which is transferred to the box and subsequently irradiated in the form of thermal electromagnetic radiation, its steps and heartbeats cause the box to vibrate, its breathing generates sound waves, and so on. As a result, the entanglement is not limited by the boundaries of the box, but propagates further, eventually encompassing the entire universe.

No realistic experiment is able to keep track of all photons and other particles that have been affected by the state of the cat and became entangled with it. But, as we discussed in the previous section, if we have lost track of at least one such particle, the state of the remaining particles is no longer defined: it is a probabilistic mixture. This explains why in real life we do not see Schrödinger cats, nor is this book in a superposition of being on the table and on the floor. Even if we are able to create such a macroscopic superposition, it will immediately entangle itself with zillions of other objects which will spread the entanglement all over the universe, making it inaccessible to experimentalists with limited capabilities. To summarize, if a superposition state contains terms that are easily distinguishable on a macroscopic level, this state is extremely fragile.

There is a further, even more surprising implication. Let us suppose the experimentalist performs measurements on multiple copies of the superposition (2.15) in the canonical basis. After the first measurement, he or she becomes a part of an entangled state which contains two terms:

$$\begin{aligned} |\Psi\rangle = \quad & \tfrac{1}{\sqrt{2}} \quad (|H\rangle \otimes |\text{experimentalist observed } H\rangle \\ & + \quad |V\rangle \otimes |\text{experimentalist observed } V\rangle). \end{aligned} \tag{2.18}$$

After the second measurement, there will be four terms:

$$\begin{aligned}|\Psi\rangle = \; & \tfrac{1}{2} \;\; (|HH\rangle \otimes |\text{observed } H \text{ in the first measurement, } H \text{ in the second measurement}\rangle \\ & + \;\; |HV\rangle \otimes |\text{observed } H \text{ in the first measurement, } V \text{ in the second measurement}\rangle \\ & + \;\; |VH\rangle \otimes |\text{observed } H \text{ in the first measurement, } H \text{ in the second measurement}\rangle \\ & + \;\; |VV\rangle \otimes |\text{observed } H \text{ in the first measurement, } V \text{ in the second measurement}\rangle)\end{aligned} \quad (2.19)$$

(where the first part of the tensor product is the state of the photons and the second part the state of the experimentalist), and so on. This superposition can be visualized as a tree-like structure, with every measurement leading to the doubling of the number of terms in the superposition, and branching of the tree (Fig. 2.3). The number of branches grows exponentially with the number of measurements: after, say, 1000 measurements, the number of terms in the superposition will equal 2^{1000}.

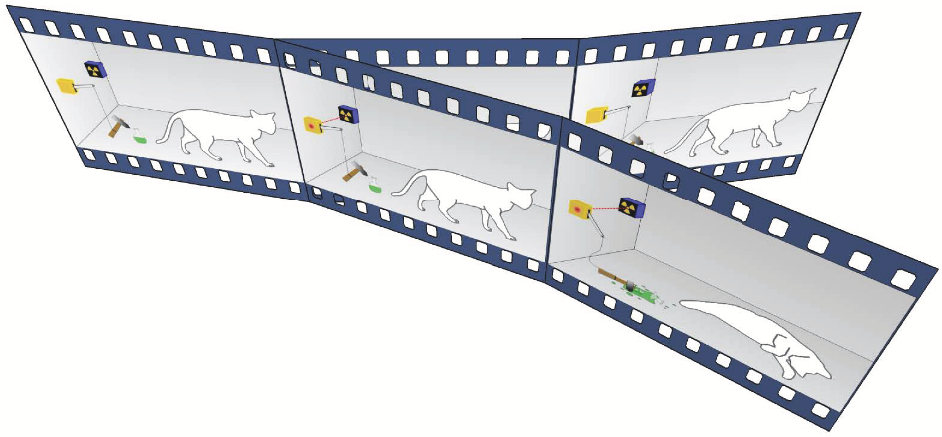

Figure 2.3: The branching of the superposition tree (from *Wikipedia*).

Let us now remember that this superposition is macroscopic. This means it encompasses the entire universe, so it is the entire universe that is in a superposition of 2^{1000} states. In fact, every time a quantum measurement takes place, the state of the universe splits into two. This includes not only measurements that physicists do, but any situation in which a microscopic superposition evolves into a macroscopic one. And such situations occur in nature uncountably often. Accordingly, the universe finds itself in an unimaginably complex superposition state. For any possible outcome for any random event or series thereof, there exists a "universe" in which it happened.

This amazing conclusion is called the *Many-world interpretation* of quantum mechanics. It was discovered by Hugh Everett in 1957 and is a natural consequence of the quantum theory if one believes it to be the ultimate, universal theory of the world. We do not have strong evidence for such belief, though. The largest objects, for which quantum superpositions have been observed, are organic molecules consisting of a few thousand atoms. One could imagine that, after exceeding a certain degree of complexity, quantum superpositions cease to exist due to fundamental reasons; in fact, some arguments stemming from general relativity seem to suggest that. But even if this is not the case, the many-world interpretation is impossible to verify, because we are unable to measure the wavefunction of the entire universe. Even if other "worlds" exists, we even theoretically have no possibility to detect them.

The last question I would like to discuss in the context of all-quantum interpretation of measurements is the explanation of Eq. (1.17) in the framework of this interpretation. If there are actually no probabilities, what is the meaning of this equation? To answer this question, let us again think of the experimentalist who measures photons in state (2.18) many times and ends up in a complex superposition state.

Although the experimentalist in each of the superposition terms cannot see other terms, he or she is aware of the full history of the measurement results that are associated with this term. Accordingly, (s)he can count how many times H and V have occurred and interpret these statistics as probabilities.

Exercise 2.27 Suppose the experimentalist performs a large number n of measurements on copies of state (2.15). Calculate the fraction of terms in the resulting giant superposition state that contain k results H and $n-k$ results V. Show this fraction to equal the probability to obtain k results H and $n-k$ results V in the framework of the standard Copenhagen interpretation

And as we can see from the exercise above, an overwhelming majority of the "worlds" will contain an approximately equal amount of the H and V events. Accordingly, experimentalists in these worlds will conclude, by looking at their result sets, that the probability to obtain either result is approximately 1/2 and the prediction of Eq. (1.17) will be validated[3].

If the amplitudes of the $|H\rangle$ and $|V\rangle$ terms in the initial photon state are not equal, the above argument will become more complicated, because different "worlds" will enter the superposition state with different amplitudes. A detailed study of this case can be found in W. H. Zurek, *Environment-Assisted Invariance, Entanglement, and Probabilities in Quantum Physics*, Physical Review Letters, volume **90**, article 120404 (2003).

2.5 Nonlocality. Einstein-Podolsky-Rosen (EPR) paradox

Definition 2.7 An obervable is an *element of physical reality* when its value can be predicted prior to measurement.

Locality principle (Local realism). If two parties are far away from each other and/or do not interact with each other, then no action by one party can change physical reality at the other.

Note 2.3 Local realism seems to be a universal principle of nature all physical theories should obey.

EPR paradox (Bohm interpretation). Suppose Alice and Bob (two remote, non-interacting parties) share two photons in the state $|\Psi^-\rangle = |HV\rangle - |VH\rangle$. Let Alice measure the polarization of her photon in the (H, V) basis. She will then remotely prepare the state $|H\rangle$ or $|V\rangle$ at Bob's. After Alice's measurement, the physical reality at Bob's station has the following features:

- If Bob chooses to measure his photon in the (H, V) basis, his measurement result can be predicted with certainty.
- If, on the other hand, Bob decides to measure in the $\pm 45°$ basis, his result would be fundamentally uncertain[4]

If Alice instead measures in the $\pm 45°$ basis, Bob's reality changes.

- If Bob chooses to measure his photon in the $\pm 45°$ basis, his measurement result can be predicted with certainty.
- If Bob decides to measure in the (H, V) basis, his result would be completely uncertain.

[3] Of course, this giant superposition will also contain terms that significantly deviate from the average: for example, there is a term with all results being H and all results being V. More generally, in the framework of the many-world interpretation, there exists, albeit with an extremely low amplitude, a world where all random events always occur with the same result. But the amplitude of this strange world, as well as all other significantly "deviant" worlds, is so minuscule that it can be neglected.

[4] "Fundamentally uncertain" means that Bob's future measurement result is not only unknown, but that quantum mechanics *prohibits* it from being known. Assuming that quantum mechanics is correct, this "lack of knowledge" is thus an integral part of physical reality associated with Bob's particle.

We see that the two physical realities described above are incompatible with each other. Therefore, according to quantum mechanics, Alice can change Bob's physical reality instantaneously and without interaction — thus violating the locality principle!

EPR's conclusion (1935). Quantum mechanics appears to be in contradiction with the fundamental laws of nature that follow from the most simple common sense. On the other hand, quantum mechanics is known to predict experimental results very well, so one cannot just say it is plain wrong. Therefore, EPR made a more careful statement, saying that that "quantum-mechanical description of reality . . . is not *complete*". According to EPR, a theory can be developed which predicts experimental results as well as QM, but does uphold local realism.

Because this "new" theory is postulated to predict the same experimental results as quantum mechanics, for the next 30 years there appeared to be no possibility to verify if EPR were correct in their hypothesis. However, in 1964 *J. Bell* proposed an experiment in which *any* local realistic theory would predict a result which is different from that predicted by QM. Specifically, he proposed an inequality *("Bell inequality")* that would hold in any local realistic theory, but is violated according to quantum mechanics.

The groups of *J.F. Clanser (1972) and A. Aspect (1982)* did this experiment and verified that QM is correct. Since then, experiments have been improving and always showed violation of the Bell inequality. However, all so far existing Bell inequality experiments contain loopholes. A conclusive experiment will be done in the next few years.

Definition 2.8 Violation of local realism by quantum mechanics is called *quantum nonlocality.*

2.6 Bell's argument

The argument consists of two parts. In the first part, we describe an experiment in terms of its superficial features, not specifying its actual construction. We will analyze this experiment using only very general physical principles, such as causality and local realism, and derive an inequality that the results of this experiment must obey. In the second part, we study a specific setup whose superficial features are consistent with those described in the first part. Using quantum mechanics, we will predict the results that turn out to violate the inequality obtained *ab initio* in the first part.

2.6.1 Local realistic argument

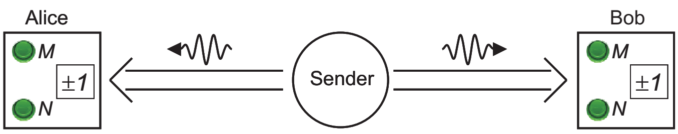

Figure 2.4: Scheme of Bell's experiment

Description of the experiment

- Each of the two remote observers, Alice and Bob, operates an apparatus shown in Fig. 2.4. The construction of the apparata is unimportant, but it is known that their front pannels have the following, identical appearance: each has two buttons marked M and N, and a display that can show either "+1" or "-1".
- Alice and Bob have no possibility to communicate with each other.
- A "sender" located about halfway between Alice and Bob sends them a pair of particles of unknown nature.

- Alice and Bob receive the particle and insert them into their "black boxes".
- Alice, Bob, and Charley simultaneously push one of the buttons on their black boxes. Each black box will then display a value of ± 1 related to the properties of the particle received. We refer to this procedure as a measurement of M_A or N_A by Alice and measurement of M_B or N_B by Bob.
- After making measurements on many particle pairs, the parties meet and discuss the results.

Derivation of the Bell inequality

Local realism implies that the measurement result obtained by each party is not affected by the button pushed by the other party. Each black box determines the value displayed for each button based on the local information at hand (i.e. the properties of the particles) and some algorithm. Therefore, the quantities M_A, N_A, M_B and N_B are elements of reality for each pair of particles once they are distributed

Consider the quantity $X = M_A(M_B - N_B) + N_A(M_B + N_B)$ for a given pair of particles. Because both M_B and N_B has a definite value of $+1$ or -1, either $(M_B - N_B)$ or $(M_B + N_B)$ must be equal to zero. Because both M_A and N_A is either $+1$ or -1, we find that X must be either $+2$ or -2: $|X| = 2$.

The measurement is repeated many times. Since, in each run, $|X| = 2$, the average value of X over a series of experiments must obey $|\langle X\rangle| \leq 2$, or

$$|\langle M_A M_B - M_A N_B + N_A M_B + N_A N_B\rangle| \leq 2. \tag{2.20}$$

This is the *Bell inequality.* Note again that it does not rely on any assumption about the physics of the particles distributed or the measurement apparata, but only on very general principles (causality + local realism). Therefore, it should hold for any experiment whose superficial description falls under that given above.

Exercise 2.28 When deriving Eq. (2.20), we assumed that the dependence of M's and N's on particle's properties is deterministic. Modify the above argument to account for possible probabilistic behavior (i.e. assuming that for a particle with a given set of properties, results $+1$ and -1 appear randomly with some probabilities).

2.6.2 Quantum argument

We now describe a specific setup that is consistent with the above description yet violates the Bell inequality. The two particles received by Alice and Bob are two photons in the Bell state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$. When Alice and Bob press their buttons, their apparata measure the following observables on their photons:
$\hat{M}_A$: eigenvalues ± 1, eigenstates $(|H\rangle, |V\rangle)$;
$\hat{M}_B$: eigenvalues ± 1, eigenstates $(|\pi/8\rangle, |\pi/8 + \pi/2\rangle)$;
$\hat{N}_A$: eigenvalues ± 1, eigenstates $(|\pi/4\rangle, |\pi/4 + \pi/2\rangle)$;
$\hat{N}_B$: eigenvalues ± 1, eigenstates $(|3\pi/8\rangle, |3\pi/8 + \pi/2\rangle)$,
where $|\theta\rangle$ denotes linear polarization at angle θ to horizontal[5].

The measurement result, which is one of the measured observables' eigenvalues (± 1), is displayed.

Our next goal is to make a quantum mechanical prediction for the measurement outcomes' statistics, so we can determine the expectation value of the observable X. This is done in the following exercise.

Exercise 2.29 Express the observables defined above as operators. Calculate the expectation values of the following operators in the state $|\Psi^-\rangle$:

a) $\hat{M}_A \otimes \hat{M}_B$;

[5] Obviously (see Ex. 1.62), $\hat{M}_A = \hat{\sigma}_z$, $\hat{N}_A = \hat{\sigma}_x$.

b) $\hat{M}_A \otimes \hat{N}_B$;

c) $\hat{N}_A \otimes \hat{M}_B$;

d) $\hat{N}_A \otimes \hat{N}_B$.

Hint: To reduce calculations, use the isotropicity of $|\Psi^-\rangle$ (Ex. 2.24) **Answer:** $-\frac{1}{\sqrt{2}}$; $\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$.

We now find that according to quantum mechanics, the expectation value of X is

$$\langle X\rangle = \langle \hat{M}_A\hat{M}_B + \hat{M}_A\hat{N}_B + \hat{N}_A\hat{M}_B + \hat{N}_A\hat{N}_B\rangle = -2\sqrt{2}, \tag{2.21}$$

which violates the Bell inequality (2.20).

Exercise 2.30 Show that no matter what buttons are pressed, in a large number of measurements each individual party will get an approximately equal number of results +1 and −1.

Exercise 2.31 Reproduce Bell's argument for other Bell states.

2.7 No-cloning theorem

Definition 2.9 *Quantum cloning* is a map in the Hilbert space $\mathbb{V}_1 \otimes \mathbb{V}_2$ ($\mathbb{V}_1 = \mathbb{V}_2$) such that, for some $|0\rangle \in \mathbb{V}_2$, $\forall\ |a\rangle \in \mathbb{V}_1$

$$|a\rangle \otimes |0\rangle \to |a\rangle \otimes |a\rangle \tag{2.22}$$

Exercise 2.32 Show that quantum cloning, as defined above, is impossible. (**Hint:** use the fact that any physically possible evolution in quantum mechanics is described by a linear operator.)

Exercise 2.33 Show that, if quantum cloning were possible, superluminal communication would also be possible. (**Hint:** use remote preparation and quantum tomography)

Note 2.4 As evidenced by Ex. 2.33, quantum mechanics and special relativity are compatible to each other. This compatibility is surprising because these theories are developed based on completely different first principles, and is not yet completely understood.

2.8 Quantum dense coding

Exercise 2.34 By sending N qubits to Bob, Alice cannot transfer more than N clasical bits of information.

Note 2.5 This restriction is a particular case of the so-called *Holevo bound* in quantum information science.

Exercise 2.35 Suppose Alice and Bob share a $|\Psi^-\rangle$. Show that by performing one of the three Pauli operators locally on her qubit, Alice can convert $|\Psi^-\rangle$ into one of the three other Bell states.

Quantum dense coding: the protocol

1. Alice and Bob share a $|\Psi^-\rangle$.
2. Alice wishes to send 2 bits of classical information to Bob. She performs $\hat{\mathbf{1}}$, $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ (according to the value of her 2 bits) on her qubit, then sends it to Bob.
3. Bob measures the 2 qubits in the Bell basis and recovers the value of Alice's two classical bits.

Alice transferred 2 bits of classical information by sending only one qubit!

2.9 Quantum teleportation

Consider two remote parties, Alice and Bob. Alice possesses one copy of a qubit state $|\chi\rangle = \alpha|H\rangle + \beta|V\rangle$ that she wishes to transfer to Bob. However, there is no direct quantum communication link along which Alice could send the qubit, and neither Alice nor Bob do have any classical information about $|\chi\rangle$. Instead, Alice and Bob share an entangled resource - one copy of the Bell state $|\Psi^-\rangle$. In addition, there exists a direct classical communication channel between the parties, using which they can exchange information. It turns out that Alice and Bob can utilize these resources to "teleport" $|\chi\rangle$, i.e. make an exact copy of this state emerge at Bob's station while collapsing Alice's original.

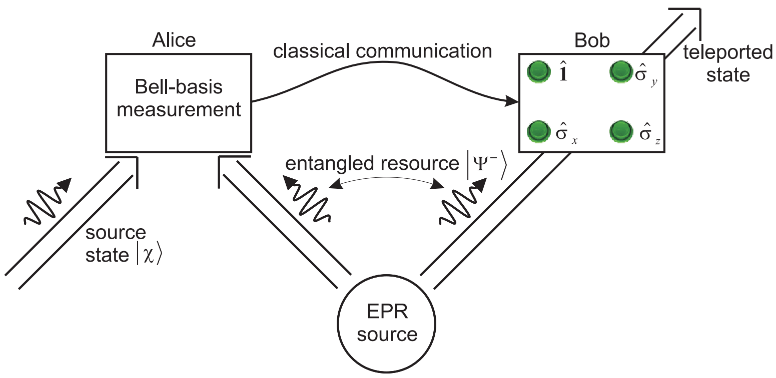

Figure 2.5: Quantum teleportation

The scheme of the quantum teleportation protocol is shown in Fig. 2.5. We say that the state $|\chi\rangle$ lives in the qubit space $\mathbb{V}_1$, and $|\Psi^-\rangle$ in $\mathbb{V}_2 \otimes \mathbb{V}_3$. The spaces $\mathbb{V}_1$ and $\mathbb{V}_2$ are localized with Alice, and $\mathbb{V}_3$ with Bob.

Exercise 2.36 Express the state $|\text{input}\rangle = |\chi\rangle \otimes |\Psi^-\rangle$ in the canonical basis of $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$. (i.e. $\{|HHH\rangle, \ldots, |VVV\rangle\}$)

Exercise 2.37 Express the canonical basis states of $\mathbb{V}_1 \otimes \mathbb{V}_2$ in the Bell basis.

Exercise 2.38 Express the state $|\chi\rangle \otimes |\Psi^-\rangle$ in the basis which is the direct product of the Bell basis in $\mathbb{V}_1 \otimes \mathbb{V}_2$ and the canonical basis in $\mathbb{V}_3$.

Exercise 2.39 Suppose Alice performs a local measurement of her fraction of state $|\text{input}\rangle$ in the Bell basis. Calculate the probability of each measurement outcome and the state onto which space $\mathbb{V}_3$ is projected.

Exercise 2.40 Alice communicates her measurement result to Bob via the classical channel. Show that with this information, Bob can convert the state of $\mathbb{V}_3$ into $|\chi\rangle$ via a local operation.

Note 2.6 In this protocol, neither Alice nor Bob obtain any classical information about the state $|\chi\rangle$ (see Ex. 2.39). This ensures its perfect transfer.

Note 2.7 Both teleportation and remote state preparation are quantum communication protocols that allow disembodied transfer of quantum information by means of an entangled state and a classical communication channel. The difference between them is that while in teleportation, Alice possesses a copy of the state she wishes she transfer, in remote state preparation she instead has full classical information about this state.

Exercise 2.41 Will teleportation work with Alice and Bob sharing

a) $|\Psi^+\rangle$

b) $|\Phi^+\rangle$

c) $|\Phi^-\rangle$

For each positive answer, determine the local operations Bob would need to perform on $\mathbb{V}_3$ after receiving classical communication from Alice.

Exercise 2.42 *(Entanglement swapping).* Consider 4 photons A, B, C, D prepared in a partially entangled state with $\left|\Psi^-_{AB}\right\rangle \otimes \left|\Psi^-_{CD}\right\rangle$. A measurement is performed on photons B and C in the Bell basis (Fig. 2.6). Determine the state of photon pair (A, D) for each possible measurement result.

Note 2.8 Entanglement swapping creates entanglement between two particles without any interaction between them.

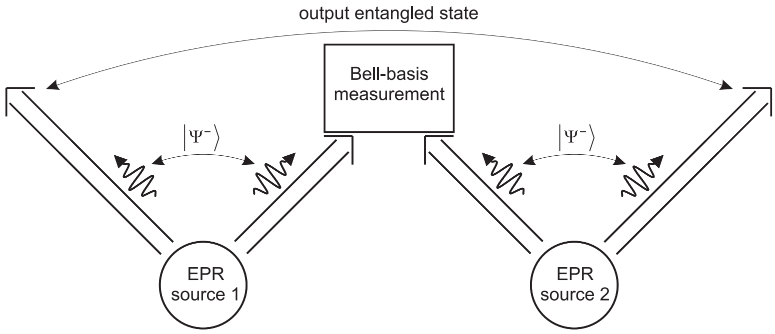

Figure 2.6: Entanglement swapping