Chapter 4

Quantum light

4.1 Quantization of the electromagnetic field

Let us consider electromagnetic field in a large evacuated box with volume $V = L_x \times L_y \times L_z$. The field inside the box can be decomposed into Fourier series

$$\vec{E}(\vec{r},t) = \sum_{\vec{k},s} e^{i\vec{k}\vec{r}} \vec{u}_{\vec{k},s}(t) + c.c; \tag{4.1a}$$

$$\vec{B}(\vec{r},t) = \sum_{\vec{k},s} e^{i\vec{k}\vec{r}} \vec{w}_{\vec{k},s}(t) + c.c, \tag{4.1b}$$

where $s = 1,2$ denotes one of the two possible polarization directions, $\vec{u}_{\vec{k},s}(t)$ and $\vec{w}_{\vec{k},s}(t)$ are the time-dependent amplitudes of the electric and magnetic fields, respectively, for a given set $(\vec{k}, s)$ [which defines a *field (plane wave) mode*].

Any arbitrary electromagnetic field configuration inside the box can be decomposed in the form (4.1). Because the box is of a finite size, it is sufficient to use a series over a discrete set of wavevectors $\vec{k} = (2\pi n_x/L_x, 2\pi n_y/L_y, 2\pi n_z/L_z)$ (where n_x, n_y, n_z are arbitrary natural numbers), rather than an integral. The behavior of the field in space is given by a periodic function whose periods in the three dimensions are equal to L_x, L_y and L_z.

In order to quantize the mode, we begin by writing its energy (Hamiltonian) H in terms of the field amplitudes. We then define the *position* x and *momentum* p that satisfy the classical *canonical equations of motion*

$$\dot{p} = -\frac{\partial H}{\partial x}; \tag{4.2a}$$

$$\dot{x} = \frac{\partial H}{\partial p}. \tag{4.2b}$$

In transition from classical to quantum mechanics, the canonical position and momentum give rise to operators whose commutator is equal to $[\hat{x}, \hat{p}] = i\hbar$.

In quantum optics, the convention is slightly different. We define the position and momentum according to $X = x/\sqrt{\hbar}, P = p/\sqrt{\hbar}$. Then the canonical equations of motion take the form

$$\dot{P} = -\frac{\partial H}{\partial X}; \tag{4.3a}$$

$$\dot{X} = \frac{\partial H}{\partial P}. \tag{4.3b}$$

and the commutator $[\hat{X}, \hat{P}] = i$.

Problem 4.1 Given that the electromagnetic energy density in vacuum equals $(\epsilon_0 E^2 + B^2/\mu_0)/2$, show that the total energy associated with a given mode in the box is given by

$$H_{\vec{k},s} = 2\epsilon_0 V |u_{\vec{k},s}(t)|^2. \tag{4.4}$$

Problem 4.2 Show that, if we define

$$X_{\vec{k},s} = \sqrt{\frac{2\epsilon_0 V}{\hbar\omega_{\vec{k}}}} \frac{u_{\vec{k},s}(t) + u_{\vec{k},s}(t)^*}{\sqrt{2}} \tag{4.5a}$$

$$P_{\vec{k},s} = \sqrt{\frac{2\epsilon_0 V}{\hbar\omega_{\vec{k}}}} \frac{u_{\vec{k},s}(t) - u_{\vec{k},s}(t)^*}{\sqrt{2}i} \tag{4.5b}$$

(where $\omega_{\vec{k}} = ck$ and the $\vec{u}$'s are treated as scalars) then

a) the Hamiltonian takes the form

$$H_{\vec{k},s} = \frac{\hbar\omega}{2}(X^2_{\vec{k},s} + P^2_{\vec{k},s}); \tag{4.6}$$

b) the canonical equations (4.3) are satisfied.

We will now treat the position and momentum associated with an electromagnetic mode as quantum operators with $[\hat{X}, \hat{P}] = i$ [the indices $(\vec{k}, s)$ will be omitted]. We see that the quantum treatment of light in vacuum is identical to that of a harmonic oscillator. In the following sections, we rederive the basic properties of important quantum states of a harmonic oscillator that are known from undergraduate quantum mechanics.

Initially, we will be working in the Schrödinger picture. This means that the operators are assumed time-independent $[\hat{X} \equiv \hat{X}(t=0), \hat{P} \equiv \hat{P}(t=0)]$, while quantum states are time-dependent. Later we will switch to the Heisenberg picture, which utilizes the opposite convention.

The position and momentum operators have eigenstates, $|X\rangle$ and $|P\rangle$, respectively, which are related according to[1]

$$\langle X| X'\rangle = \delta(X - X'); \tag{4.7a}$$

$$\langle P| P'\rangle = \delta(P - P'). \tag{4.7b}$$

The position and momentum eigenstates are related to each other via the *de Broglie wave*:

$$\langle X| P\rangle = \frac{1}{\sqrt{2\pi}} e^{iPX} \tag{4.8}$$

and thus, for an arbitrary state $|\psi\rangle$

$$\langle X| \psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \langle P| \psi\rangle\, e^{iPX} \mathrm{d}P; \tag{4.9a}$$

$$\langle P| \psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \langle X| \psi\rangle\, e^{-iPX} \mathrm{d}X. \tag{4.9b}$$

In other words, the wave functions of a given state $|\psi\rangle$ in the position and momentum representations are direct and inverse Fourier transforms of each other.

Problem 4.3 Show that

$$\langle X| \hat{P} |\psi\rangle = -i\frac{d}{dX}\psi(X); \quad \langle P| \hat{X} |\psi\rangle = i\frac{d}{dP}\psi(P). \tag{4.10}$$

Hint: $\hat{X} = \int_{-\infty}^{+\infty} X\, |X\rangle\langle X|\, \mathrm{d}X$; $\hat{P} = \int_{-\infty}^{+\infty} P\, |P\rangle\langle P|\, \mathrm{d}P$.

[1] Because these states are not normalizable, they, strictly speaking, belong not to the Hilbert space, but to the so-called *rigged Hilbert space*. See R. de la Madrid, Eur. J. Phys. **26**, 287-312 (2005) for details.

Problem 4.4 The *annihilation* operator is defined as follows:

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right); \tag{4.11}$$

The operator $\hat{a}^\dagger$ is called the *creation* operator. Show that:

a) the creation operator is

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{X} - i\hat{P}\right); \tag{4.12}$$

b) the creation and annihilation operators are not Hermitian;

c) their commutator is

$$[\hat{a}, \hat{a}^\dagger] = 1; \tag{4.13}$$

d) position and momentum can be expressed as

$$\hat{X} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right); \quad \hat{P} = \frac{1}{i\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right); \tag{4.14}$$

e) the Hamiltonian can be written as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right); \tag{4.15}$$

f) the following commutator relations hold:

$$[\hat{a}, \hat{a}^\dagger\hat{a}] = \hat{a}; \quad [\hat{a}^\dagger, \hat{a}^\dagger\hat{a}] = -\hat{a}^\dagger. \tag{4.16}$$

4.2 Fock states

Our next goal is to find the eigenvalues and eigenstates of the Hamiltonian. Because of Eq. (4.15) the latter are also eigenstates of $\hat{a}^\dagger\hat{a}$.

Problem 4.5 Suppose some state $|n\rangle$ is an eigenstate of the operator $\hat{n} = \hat{a}^\dagger\hat{a}$ [called the *(photon) number operator*] with eigenvalue n. Then

a) the state $\hat{a}\,|n\rangle$ is also an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue $n-1$;

b) the state $\hat{a}^\dagger\,|n\rangle$ is also an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue $n+1$.

Hint: Use Eq. (4.16).

Note 4.1 The above exercise shows that the states $\hat{a}\,|n\rangle$ and $\hat{a}^\dagger\,|n\rangle$ are proportional to normalized states $|n-1\rangle$ and $|n+1\rangle$, respectively. In the following, we find the proportionality coefficient.

Problem 4.6 Using $\langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle = n$, show that

a)

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,; \tag{4.17}$$

b)

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,; \tag{4.18}$$

Note 4.2 We found that, unless $n = 0$, if the state $|n\rangle$ with energy $\hbar\omega(n + 1/2)$ exists (i.e. is an element of the Hilbert space), so does the state $|n - 1\rangle$ with energy $\hbar\omega(n - 1/2)$. Similarly, states $|n - 2\rangle$, $|n - 3\rangle$ etc. must also exist. On the other hand, negative energy states are not allowed. The only way to resolve this contradiction is to assume that n *must be nonnegative integer* so the chain is broken at $n = 0$ (in which case $\hat{a}\,|0\rangle = |\text{zero}\rangle$). Furthermore, if we prove the existence of the state $|n = 0\rangle$, we automatically prove the existence of all higher energy states because

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle\,. \tag{4.19}$$

Note 4.3 Energy eigenstates of a harmonic oscillator are called *Fock* or *number* states. The state $|0\rangle$ is called the *vacuum state*. From the above results, we conclude that the Fock states have energy $\hbar\omega(n + 1/2)$, where n is a nonnegative integer.

Problem 4.7 Using $\hat{a}\,|0\rangle = 0$, calculate the wavefunction of the vacuum state in the position and momentum representations. **Hint:** use Eq. (4.10).
Answer:

$$\psi_0(X) = \frac{1}{\pi^{1/4}}e^{-X^2/2}; \quad \tilde{\psi}_0(P) = \frac{1}{\pi^{1/4}}e^{-P^2/2}. \tag{4.20}$$

Note 4.4 Energy eigenvalues are nondegenerate. Indeed, as we show in Problem 4.7, the equation $\hat{a}\,|0\rangle = 0$ has only one normalizable solution, therefore there is only one vacuum state. The higher energy Fock states are obtained from the vacuum state by applying the creation operator, and hence are also unique. As eigenstates of a Hermitian operator, Fock states form a basis.

Problem 4.8 a) By applying Eq. (4.19), calculate the wavefunctions of Fock states $|1\rangle$ and $|2\rangle$.

b)* Calculate the wavefunction of an arbitrary Fock state $|n\rangle$.

Answer:

$$\psi_n(X) = \frac{H_n(X)}{\pi^{1/4}\sqrt{2^n n!}}e^{-X^2/2}, \tag{4.21}$$

where $H_n(X)$ are the Hermite polynomials.

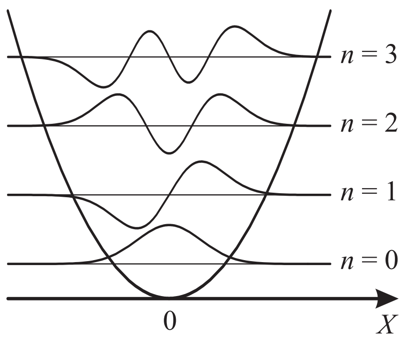

Figure 4.1: Wavefunctions of the first three energy levels of a harmonic oscillator.

Problem 4.9 For an arbitrary $|n\rangle$, calculate $\langle X\rangle$, $\langle \Delta X^2\rangle$, $\langle P\rangle$, $\langle \Delta P^2\rangle$ and verify the uncertainty principle. (**Hint:** do *not* use wavefunctions!)

Note 4.5 The vacuum state is the only minimum-uncertainty Fock state.

Problem 4.10 Find the evolution of the state $\alpha\,|0\rangle + \beta\,|1\rangle$; calculate the time dependence of $\langle X\rangle$, $\langle P\rangle$ and plot the trajectory in the phase space[2].

[2]The *phase space* is the two-dimensional space whose axes are X and P.

4.3 Coherent states

A coherent state is an important state of the harmonic oscillator, whose behavior is very similar to that of a classical oscillating particle.

Definition 4.1 A *coherent state* $|\alpha\rangle$ is an eigenstate of the annihilation operator with eigenvalue α:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,. \tag{4.22}$$

Problem 4.11 Find the decomposition of the coherent state into the number basis (make sure your answer is properly normalized).
Answer:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{4.23}$$

Note 4.6 This shows that there exists a coherent state for each complex α.

Note 4.7 If one performs an energy measurement on a coherent state, the probability to project onto a particular Fock state is given by the *Poissonian distribution*:

$$\mathrm{pr}_n = |\langle n|\,\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{4.24}$$

Problem 4.12 Find $\langle n\rangle$, $\langle\Delta n^2\rangle$ for a coherent state.
Answer: $\langle n\rangle = \langle\Delta n^2\rangle = |\alpha|^2$.

Problem 4.13 Show that in the limit of large α, the Poissonian distribution approaches Gaussian.

Problem 4.14 Find the overlap $\langle\alpha|\,\alpha'\rangle$.
Answer: $\langle\alpha|\,\alpha'\rangle = e^{-|\alpha|^2/2-|\alpha'|^2/2+\alpha'^*\alpha}$.

Problem 4.15 Do eigenstates of the creation operator exist and if so, what is their decomposition into the number basis?

Problem 4.16 Find the wavefunctions of the coherent state in the position and momentum basis. Find $\langle X\rangle$, $\langle\Delta X^2\rangle$, $\langle P\rangle$, $\langle\Delta P^2\rangle$
Answer:

$$\psi_{|\alpha\rangle}(X) = \psi_0(X-X_0)e^{iP_0X} \text{ with } \frac{X_0+iP_0}{\sqrt{2}} = \alpha. \tag{4.25}$$

Note 4.8 In coherent states, similarly to the vacuum state, the position-momentum uncertainty product takes the minimum possible value.

Problem 4.17 Find the action of the evolution operator $\exp(-i\hat{H}t/\hbar)$ upon the state $\hat{\alpha}$. Find $\langle X\rangle$ and $\langle P\rangle$ as functions of time. Plot the trajectory in the phase space.
Answer: A coherent state evolves into another coherent state with a different eigenvalue: $\exp(i\hat{H}t/\hbar)\,|\alpha\rangle = e^{-i\omega t/2}\,\left|e^{-i\omega t}\alpha\right\rangle$.

4.4 Wigner function

Consider a classical particle that is prepared multiple times with random values of position X and momentum P that follow a certain probability distribution — the *phase-space probability density* $W(X,P)$. The probability that the particle will have some certain values of position and momentum (with some tolerances) is proportional to $W(X,P)$.

Suppose now, that every time the particle is prepared, we measure the observable

$$X_\theta \equiv X\cos\theta + P\sin\theta. \tag{4.26}$$

Based on the data obtained in this measurement, we can construct a *histogram* of the experimental results, a.k.a. *marginal distribution* $\mathrm{pr}(X_\theta)$. This marginal distribution is the integral projection of the phase-space probability density on the vertical plane oriented at angle θ with respect to the vertical axis (Fig. 4.2):

$$\mathrm{pr}(X_\theta) = \int_{-\infty}^{+\infty} W(X\cos\theta - P\sin\theta, X\sin\theta + P\cos\theta)\mathrm{d}P. \tag{4.27}$$

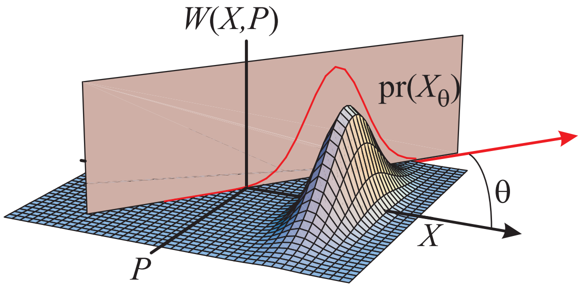

Figure 4.2: The phase-space probability density and the marginal distribution.

Now let us suppose the same measurement of $\hat{X}_\theta$ is performed on a quantum particle in state with density matrix $\hat{\rho}$ that is prepared anew after each measurement. Again, we can construct the histogram, which is related to the state according to

$$\mathrm{pr}(X_\theta) = \mathrm{Tr}(\hat{\rho}\hat{X}_\theta). \tag{4.28}$$

In the quantum domain, there can exist no phase-space probability density because, according to the uncertainty principle, the particle cannot have certain values of position and momentum at the same time. However, for every quantum state there exists a phase-space *quasi*probability density — a function $W_{\hat{\rho}}(X,P)$ for which Eq. (4.27) holds for all θ's. Without derivation, the expression for this quasiprobability density (now known as the *Wigner function*) is as follows:

$$W_{\hat{\rho}}(X,P) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iPQ}\left\langle X - \frac{Q}{2}\right|\hat{\rho}\left| X + \frac{Q}{2}\right\rangle \mathrm{d}Q. \tag{4.29}$$

This equation is called the *Wigner formula.*

Problem 4.18 Prove Eq. (4.27) for the classical case.

Problem 4.19 Use a mathematical software package to calculate the Wigner function and verify that Eq. (4.27) holds for $\theta = 0$ and $\theta = \pi/2$ for the following states:

a) vacuum state;

b) coherent state with $\alpha = 2$;

c) coherent state with $\alpha = 2e^{i\pi/6}$;

d) the single-photon state;

e) the ten-photon state;

f) $(|0\rangle + |1\rangle)/\sqrt{2}$;

g) $(|0\rangle + i\,|1\rangle)/\sqrt{2}$;

h) state with the density matrix $(|0\rangle\langle 0| + |1\rangle\langle 1|)/2$;

i) squeezed vacuum state $\psi_r(X) = \frac{1}{\pi^{1/4}\sqrt{r}} e^{-x^2/2r^2}$ with $r = 2$;

j) Schrödinger cat state $|\alpha\rangle \pm |-\alpha\rangle$ with $\alpha = 3$ and $\alpha = 1$ (please also calculate the norm for these states);

k) state with the density matrix $(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)/2$ with $\alpha = 3$.

Problem 4.20 Verify the following properties of the Wigner function:

a)
$$\iint\limits_{-\infty}^{+\infty} W_{\hat{\rho}}(X, P)\mathrm{d}X\mathrm{d}P = 1; \tag{4.30}$$

b)
$$W_{(\alpha\hat{\rho}_1+\beta\hat{\rho}_2)}(X, P)\mathrm{d}X\mathrm{d}P = \alpha W_{\hat{\rho}_1}(X, P) + \beta W_{\hat{\rho}_2}(X, P); \tag{4.31}$$

c) the state is uniquely defined by its Wigner function;

d) Wigner function of a state is real.

Note 4.9 We see that the quantum phase-space quasiprobability density inherits most of its properties from its classical counterpart. One difference of the quantum case is that the Wigner function is allowed to take on negative values (e.g. with the one-photon state). This is because the Wigner function no longer has the meaning of a probability distribution. The marginals of the Wigner function, however, do have a meaning of measurable probability densities even in the quantum case. So whenever the Wigner function has a negative "well", it must be surrounded by a positive "hill" so all its projections are nonnegative.

Note 4.10 One can define the Wigner function for any operator, rather than just density operators, by analogy with Eq. (4.29). However, the Wigner function of an arbitrary operator is not necessarily real and normalized.

Problem 4.21 Show that for any two operators $\hat{A}$ and $\hat{B}$,

$$\mathrm{Tr}(\hat{A}\hat{B}) = 2\pi \iint\limits_{-\infty}^{+\infty} W_{\hat{A}}(X, P)W_{\hat{B}}(X, P)\mathrm{d}X\mathrm{d}P \tag{4.32}$$

Note 4.11 An important consequence of the above result is that, if we know the Wigner function of a state, we can calculate its density matrix in any basis. For example, in the Fock basis:

$$\rho_{mn} = \mathrm{Tr}(\hat{\rho}\,|m\rangle\langle n|) = 2\pi \iint\limits_{-\infty}^{+\infty} W_{\hat{\rho}}(X, P)W_{|m\rangle\langle n|}(X, P)\mathrm{d}X\mathrm{d}P, \tag{4.33}$$

where $W_{|m\rangle\langle n|}(X, P)$, i.e. the Wigner function of the operator $|m\rangle\langle n|$, is easily calculated.

Problem 4.22 Show that
$$\forall X, P : W(X, P) \leq 1/\pi. \tag{4.34}$$

Hint: Develop the proof for pure states first. Use the Wigner formula and the Cauchy-Schwarz inequality. In order to generalize the result to mixed states, use Eq. (4.31).

Problem 4.23 Show that
$$W_{\hat{\rho}}(0, 0) = \frac{1}{\pi}\mathrm{Tr}(\hat{\rho}\hat{\Pi}), \tag{4.35}$$

where $\hat{\Pi} = (-1)^{\hat{n}}$ is the parity operator.
Hint: Again, start with the Wigner formula for pure states. Remember that wavefunctions of n-photon number states are of the same parity (odd or even) as number n.

Note 4.12 This result is widely employed in experimental physics in state tomography. After measuring the photon number statistics $\mathrm{pr}(n)$ of a quantum state, one finds the mean value of the parity operator $\langle\hat{\Pi}\rangle = \sum_{n=0}^{\infty}(-1)^n\mathrm{pr}(n)$ and thus the value of the Wigner function at the phase space origin. In order to determine the Wigner function at other points, one needs to perform the same measurement after applying the phase space displacement operator (see below) to the state being examined.

Note 4.13 It follows from Eq. (4.35) that for number states $W_{|n\rangle\langle n|} = (-1)^n/\pi$, which is consistent with Eq. (4.34).

4.5 Other phase-space distributions

The *Husimi function* (*Q function*) of state $\hat{\rho}$ is the convolution of the Wigner function of that state with the vacuum state Wigner function.

$$Q_{\hat{\rho}}(X,P) = \iint_{-\infty}^{+\infty} W_{\hat{\rho}}(X',P')W_{|0\rangle\langle 0|}(X'-X,P'-P)\,\mathrm{d}X'\mathrm{d}P' \tag{4.36}$$

Problem 4.24 Prove the following properties of the Q function:

a)
$$Q_{\hat{\rho}}(X,P) = \iint_{-\infty}^{+\infty} W_{\hat{\rho}}(X',P')W_{|\alpha\rangle\langle\alpha|}(X',P')\,\mathrm{d}X'\mathrm{d}P' \text{ with } \alpha = \frac{X+iP}{\sqrt{2}}; \tag{4.37}$$

b)
$$Q_{\hat{\rho}}(X,P) = \frac{1}{2\pi}\langle\alpha|\,\hat{\rho}\,|\alpha\rangle \text{ with } \alpha = \frac{X+iP}{\sqrt{2}}; \tag{4.38}$$

[**Hint:** use Eq. (4.32)]

c)
$$Q_{\hat{\rho}}(X,P) \geq 0; \tag{4.39}$$

d)
$$\iint_{-\infty}^{+\infty} Q_{\hat{\rho}}(X,P)\mathrm{d}X\mathrm{d}P = 1. \tag{4.40}$$

Problem 4.25 Calculate the Q function of the single-photon state. Compare it with the Wigner function of an equal mixture of the single photon and vacuum.

Note 4.14 This result can be generalized: the Wigner function of a quantum state that has undergone a 50% loss is identical to the state's rescaled Q function.

The *Glauber-Sudarshan function* (*P function*) of state $\hat{\rho}$ is the *de*convolution of the Wigner function of that state with the vacuum state Wigner function.

$$W_{\hat{\rho}}(X,P) = \iint_{-\infty}^{+\infty} P_{\hat{\rho}}(X',P')W_{|0\rangle\langle 0|}(X'-X,P'-P)\,\mathrm{d}X'\mathrm{d}P' \tag{4.41}$$

Note 4.15 For many states, the P function only exists as a highly singular generalized function. However, any state can be approximated, with arbitrary high fidelity, with a state that has a regular P function[3].

Problem 4.26 a) Calculate the P function of a coherent state $|\alpha\rangle$.

[3] J. R. Klauder, Phys. Rev. Lett. **16**, 534 (1966).

b) Calculate the Fourier transform of the P function of the squeezed vacuum state with $\psi_r(X) = \pi^{-1/4}r^{-1/2}e^{-x^2/2r^2}$. Verify that the P function of this state does not exist as a regular function.
Hint: Fourier transformation transforms a convolution into a product.

Problem 4.27 Prove the *optical equivalence theorem:*

$$\hat{\rho} = \iint_{-\infty}^{+\infty} P_{\hat{\rho}}(X', P')\, |\alpha\rangle\langle\alpha|\; \mathrm{d}X'\mathrm{d}P' \text{ with } \alpha = \frac{X' + iP'}{\sqrt{2}}. \tag{4.42}$$

Hint: Possible steps towards the solution are as follows:

a) express both sides of Eq. (4.42) in the coordinate basis, i.e. calculate $\langle X_1|\, \hat{\rho}|\, X_2\rangle$;

b) substitute the Wigner formula into the left-hand side of Eq. (4.41) and apply the inverse Fourier transform with respect to P to both sides of that equation;

c) verify that the results of (a) and (b) are identical.

Problem 4.28 It may appear from the optical equivalence theorem and Note 4.15 that any quantum state can be arbitrarily well approximated by a statistical mixture of coherent states. Is this correct?

4.6 Nonclassicality criteria

A quantum state is called *classical* if it can be written as a statistical mixture of coherent states. Otherwise it is called *nonclassical.* As follows from the optical equivalence theorem, classicality of a state is equivalent to its P function being positive definite[4].

If we know a quantum state exactly, we can find out if it is nonclassical by a simple calculation. In an experiment, however, only partial information about a state is often available. In such cases one can use *nonclassicality criteria* — sufficient, but not necessary conditions of nonclassical character of a state that can be verified by a measurement that is simpler than full state tomography.

Problem 4.29 The *squeezing* criterion states that a state is nonclassical if its quadrature variance for at least one phase is below that of the vacuum state (*standard quantum limit*): $\left\langle \Delta X_\theta^2 \right\rangle \geq 1/2$.

a) Verify that classical states do not satisfy the squeezing criterion.
Hint: Remember the intuitive meaning of a statistical mixture of states: each term in the mixture occurs with some probability. Show that the criterion is not satisfied by each term in the mixture, and then argue that it cannot be satisfied by the entire mixture.

b) Give an example of a state that does not satisfy the squeezing criterion, yet is nonclassical.

The *antibunching* criterion states that a state is nonclassical if its second-order coherence function is less then one:

$$g^{(2)}(0) < 1 \tag{4.43}$$

(*cf.* Problem 3.8). It is implied that each moment in time is associated with an electromagnetic mode, and all these modes have identical quantum states. In determining $g^{(2)}$, the so-called *normal ordering* of operators is used, i.e. all creation operators must precede all annihilation operators. Treating the intensity as a quantum operator, we write $I(t) \propto \hat{a}^\dagger(t)\, \hat{a}(t)$ and thus the classical formula (3.10) takes the following form in the normal order:

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(t)\, \hat{a}^\dagger(t+\tau)\, \hat{a}(t)\, \hat{a}(t+\tau)\rangle}{\langle \hat{a}^\dagger(t)\, \hat{a}(t)\rangle^2}, \tag{4.44}$$

where averaging is in the quantum sense.

An equivalent criterion is the negativity of the *Mandel parameter*

$$Q = \frac{\left\langle \Delta n^2 \right\rangle - \langle n\rangle}{\langle n\rangle} < 0. \tag{4.45}$$

[4]A simple delta function is considered positive definite for the purposes of this analysis.

Problem 4.30 For the antibunching and Mandel parameter criteria,

a) verify that these criteria are equivalent;

b) verify that these criteria are not satisfied by classical states;

c) give an example of a state that does not satisfy these criteria, yet is nonclassical.

Problem 4.31 Show that any state that has zero probability to contain exactly n photons ($\langle n|\,\hat{\rho}|\,n\rangle = 0$), for any $n \neq 0$, is nonclassical.

Note 4.16 This implies that any state with a finite number of terms in its decomposition into the Fock basis is nonclassical.

Problem 4.32 Another important nonclassicality criterion is the state's Wigner function taking on negative values somewhere in the phase space.

a) Verify that the Wigner function negativity criterion is not satisfied by classical states;

b) Give an example of a state that does not satisfy this criterion, yet is nonclassical.

Problem 4.33 Consider the following criterion: the state is nonclassical if the probability for it to contain an odd number of photons is higher than 1/2. Show that any state that satisfies this condition also satisfies the Wigner function negativity criterion.

Problem 4.34 Which of the states in Problem 4.19 are nonclassical?

Problem 4.35 The *Vogel criterion*[5] is as follows: the state is nonclassical if the Fourier transform of any of its marginal distributions decays slower than that of the vacuum state, i.e.

$$\exists\theta, k_\theta :\ \mathrm{pr}_F(k_\theta) > e^{-k_\theta^2/4}, \tag{4.46}$$

where k_θ is the Fourier variable.

a) Prove the Vogel criterion.

b)* Verify that all nonclassical states in Problem 4.19 satisfy this criterion.

c)* Find a nonclassical state that does not satisfy the Vogel criterion.

Problem 4.36 The state with Boltzmann photon number statistics

$$\hat{\rho} = (1 - e^{-\beta}) \sum_{n=0}^{\infty} |n\rangle\langle n|\, e^{-\beta n} \tag{4.47}$$

where $\beta = \hbar\omega/k_B T$ for k_B being the Boltzmann constant and T the temperature associated with the state.

a) Calculate $\langle n\rangle$ for the thermal state and verify it to obey the Bose statistics.

b) Calculate $g^{(2)}(\tau)$ for the thermal state for $\tau = 0$ and $\tau \gg \tau_c$ (i.e. the moments t and $t+\tau$ representing separate electromagnetic modes).

c) Show that the Wigner function of the thermal state is

$$W(X,P) = \frac{1}{\pi}\tanh(\beta/2)\exp\left[-(X^2+P^2)\tanh(\beta/2)\right]. \tag{4.48}$$

Hint: Calculate the Q function using Eq. (4.38) first.

[5] W. Vogel, Phys. Rev. Lett. **84**, 1849 (2000).

4.7 A few important operators

In this section, we will employ the Heisenberg picture of quantum evolution, which assumes that all operators evolve according to

$$\hat{A}(t) = e^{i(\hat{H}/\hbar)t}\hat{A}_0 e^{-i(\hat{H}/\hbar)t} \tag{4.49}$$

while all quantum states stay constant. This is in contrast to the Schrödinger picture, which assumes that quantum states evolve:

$$|\psi(t)\rangle = e^{-i(\hat{H}/\hbar)t}\,|\psi_0\rangle\,, \tag{4.50}$$

while operators are constant.

Problem 4.37 For the Heisenberg picture,

a) show that the behavior of operator expectation values $\langle\psi|\, A\, |\, \psi\rangle$ as a function of time is the same as in the Schrödinger picture;

b) show that the operator evolution can be written in the form

$$\partial_t \hat{A}(t) = \frac{i}{\hbar}[\hat{H}, \hat{A}(t)]. \tag{4.51}$$

Note 4.17 The right-hand sign of the differential equation for the evolution of the density operator in the Schrödinger picture has the opposite sign compared to Eq. (4.51): $\partial_t \hat{\rho}(t) = -i/\hbar[\hat{H}, \hat{\rho}(t)]$.

Problem 4.38 Consider a classical mechanical harmonic oscillator with the Hamiltonian $\hat{H} = \hat{p}^2/2m + k\hat{x}^2/2$. Show that the Heisenberg equation of motion (4.51) for the position and momentum is exactly the same as the classical (Hamiltonian) equation of motion under Hamiltonian equations.

Note 4.18 If the Hamiltonian contains only terms up to the second order in the position and momentum, one can show[6] that each point in the phase space evolves according to the classical equations of motion. So if the position and momentum transform according to $(\hat{X}(t), \hat{P}(t)) = F_t(\hat{X}_0, \hat{P}_0)$, (where $F_t(\cdot,\cdot)$ is an affine transformation), the Wigner function evolves in an intuitive fashion, i.e. as follows:

$$W(X, P, t) = W(F_t^{-1}(X, P), 0). \tag{4.52}$$

Problem 4.39 The *phase-space displacement* operator is given by

$$\hat{D}(X_0, P_0) = \exp(iP_0\hat{X} - iX_0\hat{P}), \tag{4.53}$$

where X_0, P_0 are real numbers.

a) White the Hamiltonian $\hat{H}$ such that the evolution $e^{-i\hat{(H/\hbar)}t_0}$ under $\hat{H}$ for time t_0 is equal to $\hat{D}(X_0, P_0)$.

b) Write the differential Heisenberg equations for the position and momentum operators under $\hat{H}$ and verify that the action of this Hamiltonian results in the transformation $\hat{X} \to \hat{X} + X_0$, $\hat{P} \to \hat{P} + P_0$.

Note 4.19 The displacement operator transforms the vacuum state into a coherent state $|\alpha\rangle$ with $\alpha = (X_0 + iP_0)/\sqrt{2}$.

Problem 4.40 The optical *phase-shift operator* is given by

$$\hat{U}(\varphi) = \exp(-i\varphi\hat{n}), \tag{4.54}$$

where φ is a real number.

[6] See, for example, W. Schleich, *Quantum Optics in Phase Space* (Wiley, 2001), Sec. 3.3.

a) Using the same approach as in the previous problem, show that the phase shift transforms the field operators as follows:

$$\hat{a} \to \hat{a}e^{-i\varphi} \tag{4.55}$$
$$\hat{a}^\dagger \to \hat{a}^\dagger e^{i\varphi} \tag{4.56}$$

b) Show that applying the phase shift operator leads to clockwise rotation of the phase space by angle φ around the origin point.

Note 4.20 The optical phase-shift operator is not the same as the quantum phase shift, given by multiplication of the state by a phase factor $e^{-i\varphi}$. The former, in contrast to the latter, has observable physical meaning. For Fock states, however, the action of the optical phase shift is equivalent to multiplication by phase factor $e^{-i\varphi n}$, i.e. does not bring about any observable modification of the state. This is the reason why Fock states have an undefined optical phase and their Wigner functions are axially symmetric.

Problem 4.41 The *single-mode (position) squeezing* operator is given by

$$\hat{S}(\zeta) = \exp[\zeta(\hat{a}^2 - \hat{a}^{\dagger 2})/2], \tag{4.57}$$

where ζ is a real number.

a) Show that the squeezing operator is associated with the following transformation:

$$\hat{a} \to \hat{a}\cosh\zeta - \hat{a}^\dagger \sinh\zeta \tag{4.58}$$
$$\hat{a}^\dagger \to \hat{a}^\dagger \cosh\zeta - \hat{a}\sinh\zeta \tag{4.59}$$
$$\hat{X} \to \hat{X}e^{-\zeta} \tag{4.60}$$
$$\hat{P} \to \hat{P}e^{\zeta}. \tag{4.61}$$

b) Calculate the Wigner function of the squeezed vacuum state $\hat{S}(\zeta)|0\rangle$.

Consider the evolution under the squeezing Hamiltonian

$$\hat{H} = i\alpha[\hat{a}^2 - (\hat{a}^\dagger)^2]/2 \tag{4.62}$$

with a real, positive α in the Schrödinger picture. Verify that the wavefunction

$$\psi(X) = \frac{1}{\pi^{1/4}\sqrt{r}}e^{-X^2/2r^2} \tag{4.63}$$

with $r = e^{-\alpha t}$ satisfies the Schrödinger equation.

Problem 4.42 The *two-mode squeezing* operator, acting on two electromagnetic modes with operators $\hat{a}_1, \hat{a}_2$ is given by

$$\hat{S}_2(\zeta) = \exp[\zeta(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger)], \tag{4.64}$$

where ζ is a real number.

a) Show that this operator is associated with the following transformation:

$$\hat{a}_1 \to \hat{a}_1\cosh\zeta - \hat{a}_2^\dagger\sinh\zeta; \tag{4.65}$$
$$\hat{a}_2 \to \hat{a}_2\cosh\zeta - \hat{a}_1^\dagger\sinh\zeta; \tag{4.66}$$
$$\hat{X}_1 \pm \hat{X}_2 \to (\hat{X}_1 \pm \hat{X}_2)e^{\mp\zeta}; \tag{4.67}$$
$$\hat{P}_1 \pm \hat{P}_2 \to (\hat{P}_1 \pm \hat{P}_2)e^{\pm\zeta}. \tag{4.68}$$
$$\tag{4.69}$$

b) Calculate the Wigner function of the two-mode squeezed vacuum state $S_2(\zeta)|0,0\rangle$. **Hint:** write the Wigner function in variables $(\hat{X}_1 \pm \hat{X}_2)/\sqrt{2}, (\hat{P}_1 \pm \hat{P}_2)/\sqrt{2}$.

Note 4.21 The mode operator transformation given by Eqs. (4.58), (4.59) or Eqs. (4.65), (4.66) is called the *Bogoliubov* transformation.

Note 4.22 Single-and two-mode squeezed vacuum states are generated, respectively, by degenerate and non-degenerate spontaneous parametric down-conversion. One can show[7], via a relatively lengthy calculation, that the decomposition of the two-mode vacuum state into the Fock basis is

$$\left\langle n_1, n_2 \middle| \hat{S}_2(\zeta) \middle| 0,0 \right\rangle = \frac{\delta_{n_1 n_2}}{\cosh\zeta}(\tanh\zeta)^{n_1}, \tag{4.70}$$

where $\delta_{n_1 n_2}$ is the Kronecker symbol, which accounts for the fact that the down-conversion photons are always generated in pairs.

Problem 4.43 Let us consider the two-mode squeezed state in the extreme case $\zeta \to \infty$. In this case we obtain the original Einstein-Podolsky-Rosen (EPR) state[8]. Suppose the modes $\hat{a}_1$ and $\hat{a}_2$ are given to two observers, Alice and Bob. Based on the results of Problem 4.42, visualize the Wigner function of the EPR state and answer the following questions.

a) Suppose Alice performs a measurement of her mode's position and obtains some result X_0. Onto which state will Bob's particle project?

b) Suppose Alice instead performs a measurement of her mode's momentum and obtains some result P_0. Onto which state will Bob's particle project?

Note 4.23 We see that by choosing to measure in the position or momentum basis, Alice can create one of two mutually incompatible physical realities associated with Bob's mode [(certain position, uncertain momentum) or (certain momentum, uncertain position)].

4.8 The beam splitter

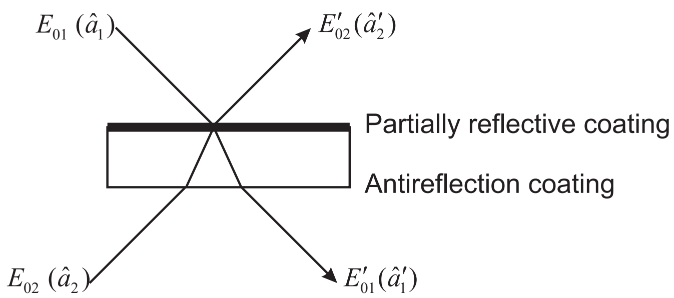

Figure 4.3: The beam splitter.

Consider two waves with amplitudes E_{01} and E_{02} overlapped on a beam splitter in matching optical modes (i.e. the reflected wave 1 is in the same mode as the transmitted wave 2 and vice versa) and generating waves E'_{01} and E'_{02}. Because the beam splitter is a linear optical device, the transformation experienced by the waves is also linear:

$$\begin{pmatrix} E'_{01} \\ E'_{02} \end{pmatrix} = \underline{B} \begin{pmatrix} E_{01} \\ E_{02} \end{pmatrix}, \tag{4.71}$$

[7]See, for example, W. Vogel, D.-G. Welsch, *Quantum Optics* (Wiley, 2006), Sec. 3.3.
[8]A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47, 777 (1935).

where

$$\underline{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{4.72}$$

is a 2×2 matrix.

Problem 4.44 Show that $\underline{B}$ is a unitary matrix.

Problem 4.45 Show that any unitary 2×2 matrix can be written in the form

$$\underline{B} = e^{i\Lambda/2} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}, \tag{4.73}$$

where all the parameters are real numbers.

Note 4.24 In Eq. (4.73),

- the first factor corresponds to a common optical (not quantum!) phase shift;
- the second factor corresponds to equal but opposite phase shifts of the output fields;
- the third factor corresponds to transmission and reflection with $r^2 + t^2$;
- the second factor corresponds to equal but opposite phase shifts of the input fields.

We see that the third factor is the only one that brings about the mixing of the modes. The remaining factors enact single-mode phase shifts and are hereafter neglected:

$$\underline{B} = \begin{pmatrix} t & -r \\ r & t \end{pmatrix}, \tag{4.74}$$

The quantities t^2 and r^2 are, respectively, the intensity *transmissivity* and *reflectivity* of the beam splitter.

If we switch to the quantum domain, the transformation of the wave amplitudes is replaced by the transformation of the field annihilation operators in the Heisenberg picture, i.e. Eq. (4.71) takes the form

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \underline{B} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}. \tag{4.75}$$

Problem 4.46 Using rule (4.52) for the Winger function transformation in the Heisenberg picture show that

a) overlapping the two modes of a two-mode squeezed vacuum state on a symmetric ($r = t = 1/\sqrt{2}$) beam splitter will result in a separable state of two single-mode squeezed vacua with the same degree of squeezing, one of which is position-squeezed and the other momentum-squeezed;

b) the reverse statement is also valid.

Our next task is to write the beam splitter transformation in the Fock basis, i.e. calculate the beam splitter output $\hat{U}\,|n_1, n_2\rangle$ if two Fock states $|n\rangle_1$, $|n\rangle_2$ are present in the input. This problem must be solved in the Schrödinger picture, whereas all our calculations so far were performed in the Heisenberg picture.

In order to switch to the Schrödinger picture, let us treat the beam splitter transformation as evolution $\hat{U} = e^{-i(\hat{H}/\hbar)t}$ that takes place under fictitious Hamiltonian $\hat{H}$ for time t. Note that matrix $\underline{B}$ is not identical to $\hat{U}$, because the form of Eq. (4.75) is clearly different from Eq. (4.49). However, we can rely on these two equations to write

$$\hat{a}_1' = \hat{U}^\dagger \hat{a}_1 \hat{U} = t\hat{a}_1 - r\hat{a}_2; \tag{4.76a}$$

$$\hat{a}_2' = \hat{U}^\dagger \hat{a}_2 \hat{U} = r\hat{a}_1 + t\hat{a}_2. \tag{4.76b}$$

Problem 4.47 Show that the beam splitter (4.74) can be associated with the Hamiltonian

$$\hat{H}_{\rm BS} = \hbar\Omega(i\hat{a}_1\hat{a}_2^\dagger - i\hat{a}_1^\dagger\hat{a}_2), \tag{4.77}$$

with $\sin\Omega t = r$.

Note 4.25 The interpretation of Hamiltonian (4.77) is that the beam splitter exchanges photons between modes 1 and 2, but preserves the total number of photons.

Problem 4.48 Calculate the Fock decomposition of the beam splitter output with input $|n_1, n_2\rangle$ as described below.

a) Show that

$$\langle m_1, m_2|\,\hat{U}\,|n_1, n_2\rangle = \frac{1}{\sqrt{m_1!m_2!}}\,\langle 0,0|\,\hat{U}^\dagger(\hat{a}_1)^{m_1}(\hat{a}_2)^{m_2}\hat{U}\,|n_1,n_2\rangle\,. \tag{4.78}$$

Hint: use the fact that the beam splitter acting upon the double vacuum state generates double vacuum: $\hat{U}\,|0,0\rangle = |0,0\rangle$

b) Use Eqs. (4.76) to verify the following:

$$\begin{aligned}&\langle m_1, m_2|\,\hat{U}\,|n_1, n_2\rangle \\ &= \sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\delta_{k_1+k_2,n_1}\delta_{n_1+n_2,m_2+m_2}\frac{(-1)^{m_1-k_1}\sqrt{n_1!n_2!m_1!m_2!}}{k_1!(m_1-k_1)!k_2!(m_2-k_2)!}t^{m_2+k_1-k_2}r^{m_1-k_1+k_2}.\end{aligned} \tag{4.79}$$

Problem 4.49 Show that if one of the beam splitter inputs is in the vacuum state, Eq. (4.79) takes a simple form

$$\hat{U}\,|n,0\rangle = \sum_{k=0}^{n} A_{nk}\,|n-k,k\rangle\,, \tag{4.80}$$

where

$$A_{nk} = \sqrt{\begin{pmatrix} n \\ k\end{pmatrix}}t^{n-k}r^k \tag{4.81}$$

Verify that the probability to detect $(k, n-k)$ photons in the two outputs could be calculated by treating photons as classical particles that have probabilities t^2 and r^2, respectively, to be transmitted or reflected.

Problem 4.50 Calculate $\hat{U}\,|1,1\rangle$. Verify the *Hong-Ou-Mandel effect*: for a symmetric beam splitter, the probability to detect one photon in each output vanishes.

4.9 The beam splitter model of absorption

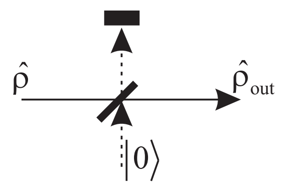

Figure 4.4: The beam splitter model of absorption.

Problem 4.51 The *beam splitter model of absorption* (Fig. 4.4) provides the means to determine the quantum state after it has undergone optical loss. The model consists of replacing the lossy channel with a beam splitter that has the same transmissivity and the vacuum state entering its other channel. The reflected mode of the beam splitter is assumed to be lost.

a) Show that if a quantum state with density matrix

$$\hat{\rho} = \sum_{m,n} \rho_{mn} |m\rangle\langle n| \tag{4.82}$$

will generate two-mode output with density matrix

$$\hat{\mathcal{P}} = \sum_{m,n} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{mj} A_{nk} |m-j, j\rangle\langle n-k, k| \,. \tag{4.83}$$

b) By taking the partial trace over the reflected mode, calculate the density matrix of the transmitted mode:

$$\hat{\rho}_{\text{out}} = \text{Tr}_2 \hat{\mathcal{P}} = \sum_{m,n} \sum_{k=0}^{n} A_{mj} A_{nk} |m-k\rangle\langle n-k| \,. \tag{4.84}$$

Note 4.26 Result (4.84) is called the *generalized Bernoulli transformation.*

Problem 4.52 Show that two coherent states with amplitudes α and β entering the beam splitter will produce two coherent states with amplitudes $B_{11}\alpha + B_{12}\beta$ and $B_{21}\alpha + B_{22}\beta$. **Hint:** Show that the state $\hat{U} |\alpha\rangle |\beta\rangle$ is an eigenstate of both $\hat{a}_1$ and $\hat{a}_2$.

Problem 4.53 Show that in order to obtain a nonclassical state at a beam splitter output, the input state must also be nonclassical.

Problem 4.54 Show that a coherent state $|\alpha\rangle$, after propagation through a loss channel with transmissivity t^2, becomes $|t\alpha\rangle$ (neglecting the phase shift).

Problem 4.55 Show that the Glauber-Sudarshan function of a state that has propagated through a loss channel with transmissivity t^2, rescales as follows:

$$P_{\text{out}}(X, P) = \frac{1}{t^2} P_{\text{in}} \left(\frac{X}{t}, \frac{P}{t} \right). \tag{4.85}$$

Hint: use the optical equivalence theorem.

Problem 4.56 Show that the Wigner function of a state that has propagated through a loss channel with transmissivity t^2, transforms as follows:

$$W_{\text{out}}(X, P) = \frac{1}{\pi t^2(1-t^2)} W_{\text{in}} \left(\frac{X}{t}, \frac{P}{t} \right) * \exp\left(-\frac{X^2 + P^2}{1-t^2} \right). \tag{4.86}$$

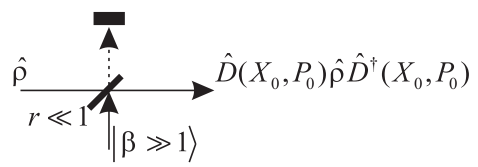

Figure 4.5: Implementation of the phase-space displacement operator with a beam splitter and a coherent state.

Note 4.27 The latter result can be interpreted as follows. The loss leads to "shrinkage" of the initial phase space, i.e. each point transforms according to $X \to tX, P \to tP$. In addition, random noise of variance $(1-t^2)/2$ (the second factor in the convolution) is added to both quadratures due to admixture of the reflected portion of the vacuum state.

Note 4.28 If $t^2 = 1/2$, Eq. (4.86) confirms the statement of Note 4.14.

Problem 4.57 Show that the phase-space displacement operator can be implemented as shown in Fig. 4.5, i.e. by transmitting the quantum state through a low-reflectivity beam splitter and shining a strong coherent state $|\beta\rangle$ into the other input channel. Show that the displacement is given by $(X_0 + iP_0)/\sqrt{2} = -\beta r$.
Hint: Calculate the P function of the output state.

4.10 Homodyne tomography

Consider a task of determining the complete information about a quantum state $\hat{\rho}$ of an optical mode by making measurements on multiple available copies of this state. To accomplish this task, it is not sufficient to perform measurements, however multiple, in a single basis, e.g. the position basis. Such measurements will provide us only with a set of probabilities $\mathrm{pr}(X) = \langle X|\,\hat{\rho}\,|X\rangle$ (i.e. the diagonal elements of the density matrix in the position basis), but no information about the phases of the wave function (the off-diagonal elements). In order to fully reconstruct the state, we need to perform measurements in multiple bases, acquiring sufficient statistics in each basis. This procedure is called *quantum state tomography.* Its general principles are applicable to any quantum system, not necessarily optical.

Problem 4.58 Show that

$$\mathrm{pr}_{F,\theta}(k_X) = 2\pi W_F(k_X \cos\theta, k_X \sin\theta), \tag{4.87}$$

where $W_F(k_X, k_P)$ is the Fourier transform of the Wigner function $W(X, P)$ of a state $\hat{\rho}$ and $\mathrm{pr}_{F,\theta}(k_X)$ is the Fourier transform of its marginal $\mathrm{pr}_\theta(X)$.

Note 4.29 We see that if we can measure the histograms $\mathrm{pr}_\theta(X)$ for all θ's, we can reconstruct the state's Wigner function and thus its density matrix (see Problem 4.20). In order to implement this in practice, we need to learn how to measure the field quadrature $X_\theta = X\cos\theta + P\sin\theta$.

This requires phase-sensitive measurement of the electromagnetic field amplitude. Given that the light field oscillates at hundreds of Terahertz, i.e. much faster than the bandwidth of any electronic circuit, such measurements cannot be performed directly. Instead, one utilizes an interference-based scheme known as *(balanced) homodyne detection* (Fig. 4.6). Here a signal mode, defined by annihilation operator $\hat{a}$ (whose state we need to measure) is overlapped with a *local oscillator* mode $\hat{a}_{\mathrm{LO}}$ containing a high-amplitude coherent state $|\alpha_{\mathrm{LO}}\rangle$ on a symmetric beam splitter. The two beam splitter output channels are detected by highly-efficient photodiodes, so each photon is converted into a photoelectron. The photocurrents are integrated in time, so the total number of photoelectrons produced by each diode corresponds to a measurement of the photon number operator $\hat{n}_1$ and $\hat{n}_2$ in the respective mode. Finally, the measurements are subtracted from each other, yielding $\hat{n}_- = \hat{n}_1 - \hat{n}_2$.

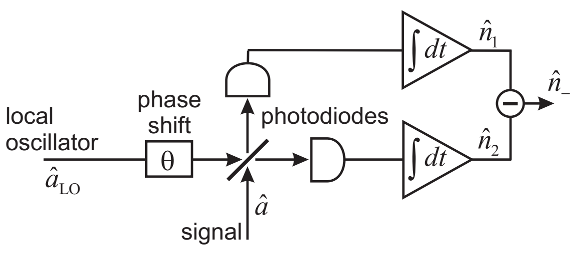

Figure 4.6: Balanced homodyne detection.

Problem 4.59 Show that

$$\hat{n}_- = \hat{a}_{\rm LO}^\dagger \hat{a} e^{i\theta} + \hat{a}_{\rm LO} \hat{a}^\dagger e^{-i\theta}, \tag{4.88}$$

where θ is the relative optical phase between the signal and the local oscillator.

Problem 4.60 One can show[9] that, for the local oscillator in a high-amplitude coherent state, the annihilation operator of the local oscillator can be approximated by its eigenvalue: $\hat{\alpha} \to \alpha_{\rm LO}$ and thus

$$\hat{n}_- = \alpha_{\rm LO}^* \hat{a} e^{i\theta} + \alpha_{\rm LO} \hat{a}^\dagger e^{-i\theta}. \tag{4.89}$$

Assuming $\alpha_{\rm LO}$ to be real, we find that the homodyne detector measures the observable that is proportional to $\hat{X}_\theta$.

Note 4.30 We see from Eq. (4.89) that the subtraction charge scales as the local oscillator amplitude, or the square root of the local oscillator intensity:

$$n_- \sim \alpha_{\rm LO} = \sqrt{N_{\rm LO}}, \tag{4.90}$$

where $N_{\rm LO}$ is the number of photons in the local oscillator. This is a macroscopic quantity, so homodyne tomography does not require detectors with single-photon sensitivity.

Problem 4.61 Suppose the beam splitter is slightly asymmetric, i.e. $t^2 - r^2 = \varepsilon \ll 1$.

a) Write the approximate expression for $\hat{n}_-$ in this situation.

b) Show that proper functioning of the detector requires $\varepsilon \ll N_{\rm LO}^{-1/2}$.

Problem 4.62 Realistic photodiodes are imperfect: they convert each photon into a photoelectron with non-unitary probability η (this quantity is called the detector's *quantum efficiency*). Such a photodiode can be modeled by a perfect photodiode preceded by an attenuator of intensity transmissivity η [Fig. 4.7(a)].

a) Write the expression for $\hat{n}_-$.

b) Show that this imperfect detector is equivalent to a detector in which the photodiodes are perfect, but the signal and local oscillator are both attenuated before the beam splitter [Fig. 4.7(b)].

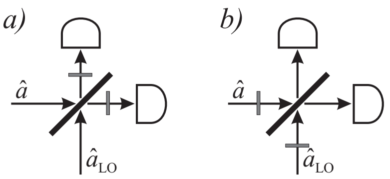

Figure 4.7: Illustration to Problem 4.62. The schemes in (a) and (b) are equivalent if both attenuators have the same transmissivity η.

So far, we have been dealing with plane-wave electromagnetic modes. This treatment is easier mathematically, but largely unpractical because plane waves have infinite extent in space and time. In order to study practically relevant modes, we remember that, according to the rules of field quantization,

$$\hat{E}(\vec{r}, t) = \sum_j \sqrt{\frac{\hbar\omega_j}{2\epsilon_0 V}} \hat{a}_j e^{i\vec{k}_j \vec{r} - i\omega_j t} + H.c, \tag{4.91}$$

[9] W. Vogel, D.-G. Welsch, *Quantum Optics* (Wiley, 2006), Sec. 6.5.4.

where we neglected the vector character of the field. As evident from this equation, a field of any arbitrary spatiotemporal shape can be obtained, using Fourier decomposition, as a linear combination of plane-wave modes:

$$\hat{A} = \sum_j \beta_j \hat{a}_j. \tag{4.92}$$

If we require that $\sum_j |\beta_j|^2 = 1$, then we have $[\hat{A}, \hat{A}^\dagger] = 1$ so annihilation operator $\hat{A}$ defines a valid optical mode.

In order to formalize this procedure, we can define a unitary transformation

$$\hat{A}_i = \sum_j U_{ij} \hat{a}_j \tag{4.93}$$

with $\hat{A}_1 = \hat{A}$, $U_{1j} = \beta_j$ and other $\hat{A}_i$'s completing the orthonormal set (obtained e.g. via the Gram-Schmidt procedure). The set $\{\hat{A}_i\}$ forms then a new basis such that one of its elements is the mode of interest, and transformation U can be treated similarly to a beam splitter.

Problem 4.63 Suppose each of the plane wave modes $\hat{a}_i$ is prepared in a coherent state $|\alpha_i\rangle$ with $\alpha_i = \alpha\beta_i$. Show that mode $\hat{A}$ is then in a coherent state $|\alpha\rangle$ while all other $\hat{A}_i$ (with $i \geq 2$) are in vacuum states.

Problem 4.64 Consider a homodyne detection experiment with the local oscillator prepared in a coherent state $|\alpha_{\rm LO}\rangle$ of mode $\hat{A}_{\rm LO} = \sum_i \beta_i \hat{a}_{{\rm LO},i}$ while all other basis modes $\hat{A}_i$ (with $i \geq 2$) are in vacuum states. Show that the subtraction charge generated by the homodyne detector is then given by

$$\hat{n}_- = \alpha^*_{\rm LO} \hat{A} e^{i\theta} + \alpha_{\rm LO} \hat{A}^\dagger e^{-i\theta}. \tag{4.94}$$

where $\hat{A} = \sum_j \beta_j \hat{a}_j$, with each $\hat{a}_j$ being associated with the mode that matches the corresponding local oscillator mode $\hat{a}_{{\rm LO},i}$.

We conclude that balanced homodyne detection measures the state of the optical mode matching that of the local oscillator.