

# Chapter 6

## Elements of atomic physics

### 6.1 Interaction picture

In the Schrödinger picture, operators are constant and states evolve:

$$|\psi_S(t)\rangle = e^{-i(\hat{H}/\hbar)t} |\psi_0\rangle. \quad (6.1)$$

In the Heisenberg picture, states are constant and operators evolve:

$$\hat{A}_H(t) = e^{i(\hat{H}/\hbar)t} \hat{A}_0 e^{-i(\hat{H}/\hbar)t}. \quad (6.2)$$

The *interaction (Dirac) picture* is halfway between the two. Suppose the Hamiltonian consists of two terms, the (typically large) *unperturbed Hamiltonian* and the (typically small) *perturbation Hamiltonian*:

$$\hat{H} = \hat{H}_0 + \hat{H}_I. \quad (6.3)$$

States in the interaction picture evolve according to

$$|\psi_I(t)\rangle = e^{i(\hat{H}_0/\hbar)t} |\psi_S(t)\rangle, \quad (6.4)$$

and operators according to

$$\hat{A}_I(t) = e^{i(\hat{H}_0/\hbar)t} \hat{A}_0 e^{-i(\hat{H}_0/\hbar)t}. \quad (6.5)$$

**Note 6.1** It may appear that Eq. (6.4) can be rewritten as  $|\psi_I(t)\rangle = e^{-i(\hat{H}_I/\hbar)t} |\psi_0\rangle$ . However, this is true only if  $\hat{H}_0$  and  $\hat{H}_I$  commute.

**Problem 6.1** Show

a) for any state:

$$\partial_t |\psi_I(t)\rangle = -\frac{i}{\hbar} \hat{V}(t) |\psi_I(t)\rangle, \quad (6.6)$$

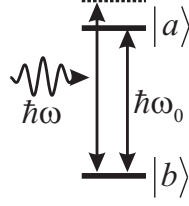
where

$$\hat{V}(t) = e^{i(\hat{H}_0/\hbar)t} \hat{H}_I e^{-i(\hat{H}_0/\hbar)t}, \quad (6.7)$$

b) for any operator,

$$\langle \psi_S(t) | \hat{A}_0 | \psi_S(t) \rangle = \langle \psi_I(t) | \hat{A}_I(t) | \psi_I(t) \rangle. \quad (6.8)$$

**Note 6.2** We see that the interaction picture is identical to the Schrödinger picture if  $\hat{H}_0 = 0$  and to the Heisenberg picture if  $\hat{H}_I = 0$ .

Figure 6.1: Two-level atom excited by a laser of frequency  $\omega$ .

## 6.2 Two-level atom

### 6.2.1 The rotating-wave approximation

Consider a two-level atom (Fig. 6.1) excited by a laser field

$$\vec{E}(t) = \vec{E}_0 e^{-i\omega t} + \vec{E}_0^* e^{i\omega t} \quad (6.9)$$

Assuming that the ground state  $|b\rangle$  has zero energy, the Hamiltonian of this system is given by

$$\hat{H} = \hbar\omega_0 |a\rangle\langle a| - \vec{E}(t)\hat{d}, \quad (6.10)$$

where

$$\hat{d} = \begin{pmatrix} \vec{d}_{aa} & \vec{d}_{ab} \\ \vec{d}_{ba} & \vec{d}_{bb} \end{pmatrix} \quad (6.11)$$

is the dipole moment operator in the  $\{|a\rangle, |b\rangle\}$  basis. It can be written as  $\hat{d} = e\hat{x}$ , where  $\hat{x}$  is the position of the electron.

**Problem 6.2** Show that for an atom positioned at the origin of the reference frame,  $\vec{d}_{aa} = \vec{d}_{bb} = 0$ .

The dipole moment operator can thus be rewritten as

$$\hat{d} = \begin{pmatrix} 0 & \vec{d} \\ \vec{d}^* & 0 \end{pmatrix}, \quad (6.12)$$

so we can thus rewrite the two-level atom Hamiltonian as

$$\hat{H} = \begin{pmatrix} \hbar\omega_0 & -\vec{E}\vec{d} \\ -\vec{E}\vec{d}^* & 0 \end{pmatrix}, \quad (6.13)$$

where  $\vec{d} = \vec{d}_{ab}$ .

**Problem 6.3** Show that the perturbation Hamiltonian in the interaction picture with<sup>1</sup>

$$\hat{H}_0 = \begin{pmatrix} \hbar\omega & 0 \\ 0 & 0 \end{pmatrix}, \quad (6.14)$$

equals

$$\hat{V}(t) = \begin{pmatrix} -\hbar\Delta & -\vec{E}\vec{d}e^{i\omega t} \\ -\vec{E}\vec{d}^*e^{-i\omega t} & 0 \end{pmatrix}, \quad (6.15)$$

where  $\Delta = \omega - \omega_0$ .

<sup>1</sup>It appears more natural to define  $\hat{H}_0 = \begin{pmatrix} \hbar\omega_0 & 0 \\ 0 & 0 \end{pmatrix}$ . However, convention (6.14) is more convenient because it leads to a constant interaction Hamiltonian (6.16) under the rotating-wave approximation.

**Problem 6.4** We now apply the *rotating-wave approximation (RWA)*: we neglect those terms in Eq. (6.15) that oscillate at an optical frequency. Show that under RWA, Eq. (6.15) takes the form

$$\hat{V}(t) = \hat{H} = \hbar \begin{pmatrix} -\Delta & -\Omega \\ -\Omega^* & 0 \end{pmatrix}, \quad (6.16)$$

where  $\Omega = \vec{E}_0 \vec{d} / \hbar$  is the *Rabi frequency*.

**Note 6.3** We have thus reduced the problem involving a fast oscillating Hamiltonian to one with a constant (or slowly varying) Hamiltonian (6.16). This significantly simplifies the calculations.

**Problem 6.5** Show that the evolution of the atomic state  $|\psi(t)\rangle = \begin{pmatrix} \psi_a(t) \\ \psi_b(t) \end{pmatrix}$ , if the atom is initially in the ground state, is given by

$$\psi_a(t) = \left( -i \frac{\Omega}{W} \sin Wt \right) e^{i\Delta t/2}; \quad (6.17a)$$

$$\psi_b(t) = \left( \cos Wt - i \frac{\Delta}{2W} \sin Wt \right) e^{i\Delta t/2} \quad (6.17b)$$

with  $W = \sqrt{\Delta^2/4 + |\Omega|^2}$  (Fig. 6.2).

**Note 6.4** This sinusoidal behavior of the state populations is referred to as *Rabi oscillations*. On resonance ( $\Delta = 0$ ), Eqs. 6.17 become

$$\psi_a(t) = -i \sin \Omega t; \quad (6.18a)$$

$$\psi_b(t) = \cos \Omega t. \quad (6.18b)$$

The quantity  $2\Omega t$  is called the *pulse area*. A pulse of area  $\pi/2$  excites the ground state into an equal superposition of the ground and excited states. A pulse of area  $\pi$  leads to inversion of the ground and excited state populations. Pulse area  $2\pi$  corresponds to a complete cycle of the atomic population, but the quantum state of the atom acquires an overall phase  $\pi$ .

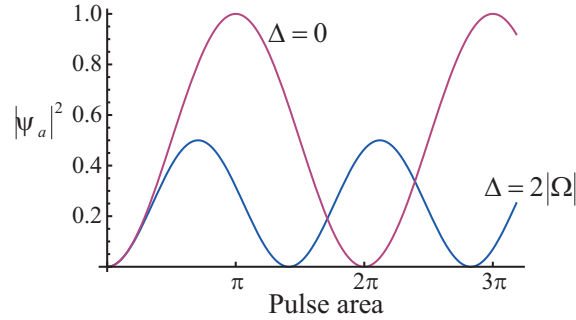


Figure 6.2: Population of the excited level as a function of the pulse area. The cases of resonant and off-resonant excitation are displayed.

**Problem 6.6** Suppose the optical frequency is slightly off resonance:  $\Delta \ll \Omega$ . Consider the following procedure:

- a pulse of area  $\pi/2$  is applied to an atom initially in the ground state;
- the atom is left alone for time  $t$ ;
- another pulse of area  $\pi/2$  is applied.

Show that the final population of the excited state behaves as  $|\psi_a|^2 = \cos^2 \Delta t/2$ . Assume that the two pulses are obtained by modulating a highly stable ( $\tau_c \gg t$ ) continuous laser field.

**Note 6.5** This phenomenon, known as *Ramsey fringes*, is widely used in metrology, particularly in atomic clocks.

### 6.2.2 Eigenstates of a two-level Hamiltonian

**Problem 6.7** Show that the eigenvalues of Hamiltonian (6.16) are given by

$$U_{1,2} = \hbar \frac{-\Delta \pm \sqrt{\Delta^2 + 4\Omega^2}}{2}. \quad (6.19)$$

Find the eigenstates (they are called the *dressed states*). Verify that for high detunings, the dressed state energies are approximated by  $(\Omega^2/\Delta, -\Delta - \Omega^2/\Delta)$ . Show that the dressed state whose energy is close to zero consists primarily of the ground state (Fig. 6.3), with the population of the excited state given by  $|\psi_a|^2 \approx \Omega^2/\Delta^2$ .

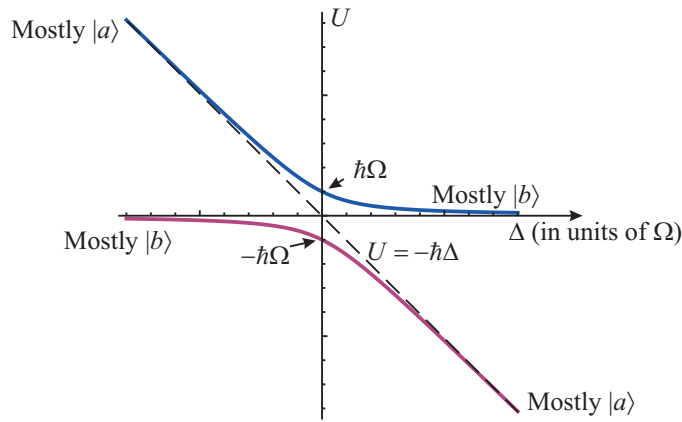


Figure 6.3: Eigenvalues of the RWA Hamiltonian.

We see that if the laser is red detuned ( $\Delta < 0$ ), the energy of the ground dressed state decreases with increasing  $\Omega$ . This means that the atom will experience a potential well of depth  $\sim \Omega^2/|\Delta|$  inside the laser beam, which is known as the *ac Stark shift* or *light shift*. It is the principle behind *dipole traps* and *optical tweezers*.

Compared to other traps for neutral atoms, the dipole trap has the advantage that it generates minimal disturbance to the atoms' quantum state. There are no resonant optical fields, nor any dc magnetic fields. On the other hand, the dipole trap requires a relatively high laser power and does not cool the atoms. Furthermore, because the trapped state contains a fraction of excited state, the atom has a chance to scatter a photon and gain kinetic energy. This occurs at a rate of

$$\Gamma_{\text{sc}} = \Gamma |\psi_a|^2 \approx \Gamma \Omega^2 / \Delta^2, \quad (6.20)$$

where  $\Gamma$  is the spontaneous decay rate.

**Problem 6.8** A dipole trap for rubidium atoms (resonance wavelength  $\lambda_0 = 780$  nm) is formed by a laser of wavelength  $\lambda = 1064$  nm of power  $P = 50$  W focused to a spot of  $r = 100$   $\mu\text{m}$  radius.

- Estimate the depth of the dipole trap (in Hz).
- Estimate the photon scattering rate given that  $\Gamma = 3.6 \times 10^7$   $\text{s}^{-1}$ .

**Hint:** Use the Bohr radius to estimate the dipole moment.

The properties of the eigenstates of the RWA Hamiltonian also give rise to a technique called *adiabatic rapid passage*. Suppose the atom is initially in the ground state. The laser is turned on far below the resonance and then slowly tuned to far above the resonance. The atom will adiabatically follow, remaining in the lower energy eigenstate (Fig. 6.3), which will change its nature from almost  $|b\rangle$  to almost  $|a\rangle$ . When the laser is turned off, the atom will likely be in the excited state.

Of course, an atom can also be transferred from  $|b\rangle$  to  $|a\rangle$  by a  $\pi$  pulse. However, this would require precise setting of the pulse duration and intensity, which can be challenging. Furthermore, the intensity of a realistic laser beam depends on the transverse position, which may become an issue if an ensemble of atoms is to be excited. Adiabatic rapid passage does not suffer from any of these shortcomings.

**Problem 6.9** Estimate the limitations on the transition times imposed by the spontaneous emission and the requirement of adiabaticity.

### 6.2.3 Master equations

**Problem 6.10** Show that the evolution of the atomic density matrix  $\hat{\rho} = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$  in the interaction picture, in the absence of spontaneous decay, assuming the Rabi frequency to be real, is given by

$$(\partial_t \hat{\rho})_{\text{int}} = \begin{pmatrix} i\Omega(\rho_{ba} - \rho_{ab}) & i\Delta\rho_{ab} + i\Omega(\rho_{bb} - \rho_{aa}) \\ -i\Delta\rho_{ba} + i\Omega(\rho_{aa} - \rho_{bb}) & i\Omega(\rho_{ab} - \rho_{ba}) \end{pmatrix}. \quad (6.21)$$

**Hint:** use Eq. (A.4).

**Problem 6.11** In order to derive the effect of the spontaneous emission on the density matrix, suppose the atom in a pure state  $\psi_a |a\rangle + \psi_b |b\rangle$  interacts with the “reservoir” — a set of electromagnetic modes initially in the vacuum state. The interaction for a short time  $\delta t$  results in the following state evolution:

$$(\psi_a |a\rangle + \psi_b |b\rangle)_{\text{atom}} \otimes |0\rangle_{\text{res}} \rightarrow \psi_a (\sqrt{1-\delta p} |a\rangle_{\text{atom}} \otimes |0\rangle_{\text{res}} + \sqrt{\delta p} |b\rangle_{\text{atom}} \otimes |1\rangle_{\text{res}}) + \psi_b |b\rangle_{\text{atom}} \otimes |0\rangle_{\text{res}}, \quad (6.22)$$

where  $|n\rangle_{\text{res}}$  denotes the state of the reservoir containing  $n$  photons (distributed over multiple modes) and  $p$  is a small number which gives the probability that the excited state will emit the photon during  $\delta t$ .

a) Take a partial trace over the reservoir and write the density matrix  $\hat{\rho}(\delta t)$  of the atom after the interaction with the reservoir.

b) Show that if the atom’s initial density matrix is given by  $\hat{\rho}(0) = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$  then

$$\hat{\rho}(\delta t) = \begin{pmatrix} (1-\delta p)\rho_{aa} & (1-\delta p/2)\rho_{ab} \\ (1-\delta p/2)\rho_{ba} & (\delta p)\rho_{aa} + \rho_{bb} \end{pmatrix} \quad (6.23)$$

c) Show that, if the probability for the excited state to emit the photon is given by  $\delta p = \Gamma \delta t$ , then the time derivative of the density matrix due to spontaneous emission is given by<sup>2</sup>

$$(\partial_t \hat{\rho})_{\text{spont}} = \begin{pmatrix} -\Gamma\rho_{aa} & -\Gamma/2\rho_{ab} \\ -\Gamma/2\rho_{ba} & \Gamma\rho_{aa} \end{pmatrix} \quad (6.24)$$

**Problem 6.12** Find the *steady state* density matrix of the two-level atom, which satisfies the equation

$$\partial_t \hat{\rho} = (\partial_t \hat{\rho})_{\text{int}} + (\partial_t \hat{\rho})_{\text{spont}} = 0. \quad (6.25)$$

**Hint:** the easiest way to solve this system of four linear equations is by expressing everything through  $\rho_{bb} - \rho_{aa}$ .

**Answer:**

$$\hat{\rho} = \begin{pmatrix} \frac{\Omega^2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} & \frac{i\Omega\Gamma/2 - \Omega\Delta}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} \\ \frac{-i\Omega\Gamma/2 - \Omega\Delta}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} & \frac{\Gamma^2/4 + \Delta^2 + \Omega^2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2} \end{pmatrix} \quad (6.26)$$

<sup>2</sup>This result can be obtained more rigorously using the *Lindblad formalism*. For details, see Preskill lecture notes, <http://theory.caltech.edu/~preskill/ph219/>, Sec. 3.5.

**Problem 6.13** Show that in the limit of weak excitation ( $\Omega \ll \Gamma$ ), to the first order in  $\Omega$ , Eq. (6.26) takes the form

$$\hat{\rho} = \begin{pmatrix} 0 & \frac{i\Omega}{\Gamma/2 - i\Delta} \\ \frac{-i\Omega}{\Gamma/2 + i\Delta} & 1 \end{pmatrix} \quad (6.27)$$

**Problem 6.14** This result can be obtained in an easy way using the semi-rigorous *stochastic wavefunction approach*, which is typically applicable when most of the atomic population is in a single energy eigenstate. The idea is to approximate, to the second order in  $\Omega$ , the state of the atom as a pure state:  $\hat{\rho} = |\psi\rangle\langle\psi|$ , where  $|\psi\rangle = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}$  with  $\psi_b \equiv 1$ . We then look for the steady state solution of the equation

$$\partial_t |\psi\rangle = -\frac{i}{\hbar} \hat{V}(t) |\psi\rangle - \frac{\Gamma}{2} \begin{pmatrix} \psi_a \\ 0 \end{pmatrix}, \quad (6.28)$$

where the second term is responsible for the spontaneous decay.

- Find the steady state solution of Eq. (6.28) taking into account only terms up to the first order in  $\Omega$ .
- Verify that your solution is consistent with density matrix (6.27) up to the first order in  $\Omega$ .

**Problem 6.15** Consider an atomic gas of number density  $N$  excited by electromagnetic wave (6.9). Calculate the expectation value of the dipole moment of an individual atom (assume  $\vec{d}$  real). Calculate the polarization of the gas as a function of time. Show that the first-order susceptibility is given by

$$\chi(\omega) = \frac{Nd^2}{\hbar\epsilon_0} \frac{\rho_{ab}}{\Omega} \stackrel{\Omega \ll \Gamma}{\approx} \frac{Nd^2}{\hbar\epsilon_0} \frac{i}{\Gamma/2 - i\Delta}. \quad (6.29)$$

**Hint:** do not forget that the dipole moment operator (6.12) is written in the Schrödinger picture. Since we are working in the interaction picture, we must account for its evolution.

**Problem 6.16** Compare Eq. (6.29) with classical susceptibility (1.30) obtained from the classical theory of dispersion for the hydrogen atom. To estimate  $d$ , use  $|\langle \hat{x} \rangle| = a_0$ , where  $a_0 = 4\pi\epsilon_0\hbar^2/(e^2m)$  is the Bohr radius, and use the Rydberg energy  $\text{Ry} = \hbar^2/(2ma_0^2)$  to estimate the frequency.

**Problem 6.17** a) Show that in the limit of low number density ( $\chi \ll 1$ ), the absorption index is given by

$$\alpha = \frac{Nd^2 \omega}{\hbar\epsilon_0 c} \frac{\Gamma/2}{\Gamma^2/4 + \Delta^2 + 2\Omega^2}. \quad (6.30)$$

- Show that Eq. (6.30) can be written in the form

$$\alpha(I) = \frac{\alpha(I=0)}{1 + I/I_{\text{sat}}}, \quad (6.31)$$

where  $I_{\text{sat}}$  is the *saturation intensity* such that the corresponding Rabi frequency satisfies  $\Omega_{\text{sat}} = \Gamma/2\sqrt{2}$ .

- Use the *Weisskopf-Wigner formula* (derived below)

$$\Gamma = \frac{d^2\omega^3}{3\pi\epsilon_0\hbar c^3} \quad (6.32)$$

to show that on resonance, in the weak excitation limit,

$$\alpha = N \frac{3\lambda^2}{2\pi}, \quad (6.33)$$

where  $\lambda$  is the optical wavelength.

**Note 6.6** Result (6.33) implies that the absorption cross-section of an atom on resonance is given by  $\sigma = 3\lambda^2/2\pi$ . This expression is remarkable because it exhibits no dependence on the atomic dipole moment.

### 6.2.4 Einstein coefficients

Consider an ensemble of two-level atoms in a reservoir containing background radiation of energy density  $\rho(\omega)$  [energy per unit volume per unit frequency, measured in  $\text{J}/(\text{m}^3\text{s}^{-1})$ ]. The densities of atoms in the excited and ground state are  $N_a$  and  $N_b$ , respectively. There is no quantum coherence between the energy eigenstates. The atoms will undergo the following processes.

- Spontaneous emission, which is characterized by the following rate equation:

$$(\dot{N}_a)_{\text{spont}} = -AN_a, \quad (6.34)$$

where  $A \equiv \Gamma$  is the spontaneous emission rate.

- Absorption:

$$(\dot{N}_a)_{\text{abs}} = B_{ba}N_b\rho(\omega_0), \quad (6.35)$$

where  $B_{ba}$  is a proportionality coefficient, proportional to the integral of the absorption over the atomic line (we neglect the variation of  $\rho(\omega)$  over this line).

- *Stimulated emission (amplification)*, a phenomenon we have not yet studied, which is inverse to absorption: affected by an electromagnetic wave, an atom in the excited state may undergo transition to the ground state, emitting a photon coherent with this wave. This effect can be obtained by performing analysis similar to the previous section, but in the presence of incoherent “pumping” from  $|b\rangle$  to  $|a\rangle$ , leading to a negative absorption index (6.30). The rate of the stimulated emission is given by

$$(\dot{N}_a)_{\text{stim}} = -B_{ab}N_a\rho(\omega_0), \quad (6.36)$$

where  $B_{ab}$  is another proportionality coefficient.

Quantities  $A$ ,  $B_{ba}$  and  $B_{ab}$  are determined exclusively by the atomic properties and are called the *Einstein coefficients*. In the following, we establish two universal relations among these coefficients.

**Problem 6.18** Let the radiation background be due to blackbody thermal radiation. It is then described by the Planck formula:

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}, \quad (6.37)$$

$T$  being the temperature. Suppose the atom is of the same temperature, i.e.

$$N_a/N_b = e^{-\hbar\omega_0/k_B T}. \quad (6.38)$$

Then the atom is in thermal equilibrium with this radiation, i.e.

$$\dot{N}_a = (\dot{N}_a)_{\text{spont}} + (\dot{N}_a)_{\text{abs}} + (\dot{N}_a)_{\text{stim}} = 0. \quad (6.39)$$

- a) Assume the temperature to be very high. Show that, in order for Eq. (6.39) to satisfy, we must have

$$B_{ba} = B_{ab} \quad (6.40)$$

- b) Substitute the above result into Eq. (6.39) for a finite temperature to show that

$$\frac{A}{B_{ab}} = \frac{\hbar\omega_0^3}{\pi^2c^3}. \quad (6.41)$$

**Note 6.7** Equation (6.41) is remarkable in that it is obtained using an argument from a seemingly unrelated field of physics (thermodynamics). Interestingly, stimulated emission was not known when Einstein derived his result. Einstein introduced stimulated emission (6.36) in order to satisfy the thermal equilibrium condition.

**Problem 6.19** Consider weak thermal radiation of density  $\varrho$  inside a gas of atoms of number density  $N_{\text{at}}$ .

- a) Show that the photon number density [photons/m<sup>3</sup>] in frequency interval  $d\omega$  is given by  $dN_{\text{phot}} = \varrho d\omega / (\hbar\omega)$ .
- b) Show that the loss of photons [photons/(m<sup>3</sup>s)] due to absorption is given by

$$(\dot{N})_{\text{abs}} = \frac{1}{3}c \int_{-\infty}^{+\infty} N_{\text{phot}}\alpha(\omega)d\omega \stackrel{(6.30)}{=} \frac{1}{3} \frac{\pi\varrho N d^2}{\epsilon_0 \hbar^2} \quad (6.42)$$

**Hint:** the factor of 1/3 comes because the thermal radiation is omnidirectional and randomly polarized, whereas the atomic dipole vector  $\vec{d}$  associated with a particular transition has a definite direction.

- c) Use Eq. (6.35) to find the Einstein  $B$  coefficient.
- d) Use Eq. (6.41) to derive the Weiskopf-Wigner formula (6.32).

### 6.3 Three-level atom

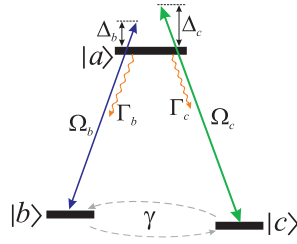


Figure 6.4: A  $\Lambda$ -type atom.

Consider an atom with a  $\Lambda$ -shaped energy level structure, as shown in Fig. 6.4. There are two ground levels  $|b\rangle$ ,  $|c\rangle$  and one excited level  $|a\rangle$ . Spontaneous emission rates from  $|a\rangle$  into  $|b\rangle$  and  $|c\rangle$  are  $\Gamma_b$  and  $\Gamma_c$ , respectively. There are two electromagnetic fields: the *control* field with Rabi frequency  $\Omega_c$  coupling  $|c\rangle$  with  $|a\rangle$  with optical frequency  $\omega_c$  and detuning  $\Delta_c$  and the *signal* field with Rabi frequency  $\Omega_b$  coupling  $|b\rangle$  with  $|a\rangle$  with optical frequency  $\omega_b$  and detuning  $\Delta_b$ . There is ground state decoherence manifesting itself as decay of the matrix element  $\rho_{bc}$  with rate  $\gamma \ll \Gamma_b, \Gamma_c$ .

**Problem 6.20** Taking  $|b\rangle$  as the zero energy state and assuming that the unperturbed Hamiltonian is given by  $\hat{H}_0 = \hbar\omega_b |a\rangle\langle a| + \hbar(\omega_b - \omega_c) |c\rangle\langle c|$ , show that the perturbation Hamiltonian in the interaction picture, rotating-wave approximation is given by

$$\hat{V} = \begin{pmatrix} -\Delta_b & -\Omega_b & -\Omega_c \\ -\Omega_b^* & 0 & 0 \\ -\Omega_c^* & 0 & -\Delta_b + \Delta_c \end{pmatrix} \quad (6.43)$$

**Problem 6.21** Show that if the *two-photon detuning*  $\Delta_b - \Delta_c$  is zero, the *dark state*

$$|\psi_{\text{dark}}\rangle = \Omega_c |b\rangle - \Omega_b |c\rangle \quad (6.44)$$

is an eigenstate of the interaction Hamiltonian.

The dark state is interesting because it has no excited state component. If the ground state decoherence is negligible, an atom prepared in this state can remain in it permanently and exhibit no absorption. This serves as a basis for a variety of interesting phenomena.

- *Coherent population trapping (CPT)*. When the two fields are tuned to a two-photon resonance, the absorption greatly reduces due to emergence of the dark state.
- Electromagnetically-induced transparency (see below).
- *Stimulated Raman adiabatic passage (STIRAP)* is a method to adiabatically transfer population between the ground levels. Suppose the atom is prepared in  $|b\rangle$  and the following manipulations are performed.
  - The field coupling  $|a\rangle$  and  $|c\rangle$  is applied. The atom is in the dark state  $|b\rangle$ .
  - The second field coupling  $|a\rangle$  and  $|b\rangle$  is slowly turned on. The atom remains in the dark state, which adiabatically changes to superposition (6.44).
  - The field between  $|a\rangle$  and  $|c\rangle$  is slowly turned off. The new dark state is  $|c\rangle$ .

In this way, the atom is transferred from  $|b\rangle$  to  $|c\rangle$ . Note that the optical pulse sequence is *counterintuitive*: the field that is initially turned on does not interact with the atom.

**Problem 6.22** *Electromagnetically-induced transparency (EIT)* is an effect related to CPT, but in EIT the signal field is very weak: its Rabi frequency is much smaller than all other Rabi frequencies and spontaneous decay rates. In this case, most of the atomic population is optically pumped into  $|b\rangle$ , so the stochastic wavefunction approximation can be used.

- a) Write the Schrödinger equation for the amplitudes  $\psi_a$  and  $\psi_c$ . **Hint:** Because, in the stochastic wavefunction approximation,  $\rho_{cb} = \psi_c \psi_b^* = \psi_c$ , we can treat the ground state decoherence as follows:  $(\dot{\psi}_c)_{\text{dec}} = -\gamma \psi_c$ .
- b) Find the steady state amplitudes.
- c) Show that the atomic medium's susceptibility with respect to the signal field is given by

$$\chi = \frac{N d_b^2}{\hbar \epsilon_0} \frac{\Delta_b - \Delta_c + i\gamma}{|\Omega_c|^2 - (\Delta_p - \Delta_c + i\gamma)(\Delta_b + i\frac{\Gamma_b + \Gamma_c}{2})} \quad (6.45)$$

**Hint:** use the first equality of Eq. (6.29).

- d) Show that for  $\Omega_c \ll \Gamma$ ,  $\Delta_c = 0$ ,  $\gamma = 0$ , the absorption spectrum of an EIT system exhibits a dip of FWHM given by

$$\text{FWHM} = 4|\Omega_c|^2 / (\Gamma_b + \Gamma_c) \quad (6.46)$$

**Note 6.8** The EIT window is a consequence of the dark state emerging at the two-photon resonance.

**Problem 6.23** Show that the group velocity in the center of the EIT window, in the approximation  $\Omega_c \ll \Gamma$ ,  $\Delta_c = 0$ ,  $\gamma = 0$  is given by

$$\frac{1}{v_{\text{gr}}} = \frac{1}{c} + \frac{N d_b^2 \omega}{2c \hbar \epsilon_0 |\Omega_c|^2}. \quad (6.47)$$

Calculate the group velocity in rubidium ( $\lambda = 795$  nm) if  $\Omega_c = 10$  kHz and  $N = 10^{12}$  cm<sup>-3</sup>.

**Problem 6.24** Show analytically that the steady state solution of Hamiltonian (6.43) in the limit  $|\Delta_c| \gg \Gamma$  leads to an additional (*Raman*) absorption line [Fig. 6.5(e)] centered at the ac Stark shifted two-photon resonance,  $\Delta_b = \Delta_c \pm |\Omega_c|^2 / \Delta_c$  (positive and negative sign corresponding to positive and negative  $\Delta_c$ , respectively) and of FWHM

$$\Gamma_{\text{Raman}} = \Gamma \frac{|\Omega_c|^2}{\Delta_c^2}. \quad (6.48)$$

The stochastic wavefunction approximation can be used.

**Problem 6.25** Show that the evolution of a three-level atom under Hamiltonian (6.43) in the limit  $|\Delta_c| \gg \Gamma$  is similar to evolution of a two-level system consisting of just the ground levels with Rabi frequency

$$\Omega_{\text{Raman}} = \frac{\Omega_c^* \Omega_b}{\Delta_c} \quad (6.49)$$

and the decay rate from  $|c\rangle$  given by Eq. (6.48). The stochastic wavefunction approximation can be used.

**Hint:** one of the evolution equations is of the form

$$\dot{\psi}_a = i\Delta_b \psi_a + i\Omega_b \psi_b + i\Omega_c \psi_c - i(\Gamma/2)\psi_a \quad (6.50)$$

You can assume that  $\psi_a$  consists of a constant term  $(\psi_a)_s$  and a quickly varying term  $(\psi_a)_q$ . We then have

$$i\Delta_b (\psi_a)_s + i\Omega_b \psi_b + i\Omega_c \psi_c - i(\Gamma/2)(\psi_a)_s = 0. \quad (6.51)$$

The quickly varying term averages to zero, so we can simply neglect it and assume that  $\psi_a = (\psi_a)_s$ , which can be readily found by solving Eq. (6.51). This procedure is called *adiabatic elimination* and is commonly used in atomic physics.

**Problem 6.26** Suppose the Rabi frequencies vary in time, but slowly. Substitute the solution of Eq. (6.51) into Eq. (6.50) and determine the conditions on  $T$ , the characteristic time scale of the variations of the Rabi frequencies.

## 6.4 Heisenberg picture of atomic transitions

**Problem 6.27** Define atomic transition operators

$$\hat{\sigma}_{H,ij}(t=0) = |i\rangle\langle j| \quad (6.52)$$

where  $i$  and  $j$  denote energy levels  $a$  or  $b$ . Write the differential equation (4.51) for the evolution of these operators in the Heisenberg picture in terms of these operators under Hamiltonian (6.13)

$$\hat{H} = \hbar\omega_0 \hat{\sigma}_{H,aa}(0) - Ed \hat{\sigma}_{H,ab}(0) - Ed^* \hat{\sigma}_{H,ba}(0). \quad (6.53)$$

Verify consistency with Eq. (6.21) (use the fact that, e.g.,  $\rho_{ab} = \text{Tr}[\hat{\rho} \hat{\sigma}_{H,ba}]$ ).

**Hint:** It is easy to see that, for example,  $[\hat{\sigma}_{H,aa}(0), \hat{\sigma}_{H,ab}(0)] = \hat{\sigma}_{H,ab}(0)$ . However, the transition operators evolve in time, so, generally speaking,  $\hat{\sigma}_{H,ij} = |i\rangle\langle j|$  only at  $t=0$ , and one may argue that the simple commutation relations between the  $\hat{\sigma}$ 's are no longer valid for  $t \neq 0$ . Fortunately, this is not the case. For example,

$$\begin{aligned} & [\hat{\sigma}_{H,aa}(t), \hat{\sigma}_{H,ab}(t)] \quad (6.54) \\ &= e^{i\hat{H}t/\hbar} \hat{\sigma}_{H,aa}(0) e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar} \hat{\sigma}_{H,ab}(0) e^{-i\hat{H}t/\hbar} - e^{i\hat{H}t/\hbar} \hat{\sigma}_{H,ab}(0) e^{-i\hat{H}t/\hbar} e^{i\hat{H}t/\hbar} \hat{\sigma}_{H,aa}(0) e^{-i\hat{H}t/\hbar} \\ &= e^{i\hat{H}t/\hbar} \hat{\sigma}_{H,aa}(0) \hat{\sigma}_{H,ab}(0) e^{-i\hat{H}t/\hbar} - e^{i\hat{H}t/\hbar} \hat{\sigma}_{H,ab}(0) \hat{\sigma}_{H,aa}(0) e^{-i\hat{H}t/\hbar} \\ &= e^{i\hat{H}t/\hbar} [\hat{\sigma}_{H,aa}(0), \hat{\sigma}_{H,ab}(0)] e^{-i\hat{H}t/\hbar} \\ &= e^{i\hat{H}t/\hbar} \hat{\sigma}_{H,ab}(0) e^{-i\hat{H}t/\hbar} = \hat{\sigma}_{H,ab}(t) \end{aligned}$$

Furthermore, because  $e^{i\hat{H}(0)t/\hbar} \hat{H}(0) e^{-i\hat{H}(0)t/\hbar} = \hat{H}(0)$ , we can write the Hamiltonian (6.53) as

$$\hat{H}(t) = \hat{H}(0) = -\hbar\omega_0 \hat{\sigma}_{H,aa}(t) - Ed \hat{\sigma}_{H,ab}(t) - Ed^* \hat{\sigma}_{H,ba}(t). \quad (6.55)$$

Thus the evolution equations do not change with time.

**Problem 6.28** Let us introduce the ‘‘slowly-varying operator picture’’, in which, in contrast to the interaction picture, operators vary slowly:

$$\hat{A}_{SV}(t) = e^{i(\hat{H}_0 + \hat{H}_I)t} e^{-i\hat{H}_0 t} \hat{A}_0 e^{i\hat{H}_0 t} e^{-i(\hat{H}_0 + \hat{H}_I)t}, \quad (6.56)$$

where  $\hat{H}_0 = \hat{H}_0(0)$  and  $\hat{H}_I = \hat{H}_I(0)$  are the unperturbed and perturbation Hamiltonians, respectively.

- a) Write the explicit expression for the state  $|\psi_{SV}(t)\rangle$  in the slowly-varying operator picture.  
 b) Show that, with the unperturbed and perturbation Hamiltonians of a two-level atom, we have

$$\begin{aligned}\hat{\sigma}_{SV,aa} &= \hat{\sigma}_{H,aa}, \\ \hat{\sigma}_{SV,ab} &= \hat{\sigma}_{H,ab}e^{-i\omega t}, \\ \hat{\sigma}_{SV,ba} &= \hat{\sigma}_{H,ba}e^{i\omega t}, \\ \hat{\sigma}_{SV,bb} &= \hat{\sigma}_{H,bb}.\end{aligned}\tag{6.57}$$

- c) Show that, for any arbitrary operator,

$$\partial_t \hat{A}_{SV} = \frac{i}{\hbar} [\hat{U}, \hat{A}],\tag{6.58}$$

where

$$\hat{U} = e^{i(\hat{H}_0 + \hat{H}_I)t} \hat{H}_I e^{-i(\hat{H}_0 + \hat{H}_I)t}.\tag{6.59}$$

- d) Show that, for a two-level atom,

$$\hat{U} = -\hbar\Delta\hat{\sigma}_{SV,aa}(t) - Ed\hat{\sigma}_{SV,ab}(t)e^{i\omega t} - Ed^*\hat{\sigma}_{SV,ba}(t)e^{-i\omega t},\tag{6.60}$$

i.e. is identical to the perturbation Hamiltonian (6.15) in the interaction picture.

**Hint:** observe that the perturbation Hamiltonian (6.59) is simply  $\hat{H}_I(t)$  that has evolved under the Heisenberg picture. It can hence be expressed in terms of  $\hat{\sigma}_{H,ij}(t)$ , which, in turn, are expressed via the  $\hat{\sigma}_{SV}$ 's using (6.57).

- e) Write the differential equation (6.58) explicitly for the four  $\hat{\sigma}_{SV}$ 's.

**Note 6.9** The relatively simple form of Eqs. (6.57) is due to a diagonal form of the unperturbed Hamiltonian  $\hat{H}_0$ . In this case, the difference between the Heisenberg picture and the slowly-varying operator picture is simple phase factors.

**Note 6.10** We will be using the slow-varying operator picture for the remainder of this section.

**Note 6.11** Eliminating the quickly-varying terms in the perturbation Hamiltonian (6.60), we write

$$\hat{U} = -\hbar\Delta\hat{\sigma}_{SV,aa}(t) - E^+\hat{\sigma}_{ab}(t)e^{i\omega t} - E^-d^*\hat{\sigma}_{ba}(t)e^{-i\omega t}.\tag{6.61}$$

This is interpreted as follows: the positive-frequency term of the field, which consists of photon annihilation operators, couples to  $\sigma_{ab}$ , a transition that excites the atom, and vice versa.

**Problem 6.29** Introduce the slowly-varying position-dependent atomic operator

$$\hat{\sigma}_{ba}(z, t) = e^{-ikz} \frac{1}{N\delta V} \sum_{\text{atom } m \in \delta V} \hat{\sigma}_{m,ba}(t),\tag{6.62}$$

i.e. the atomic transition operator averaged over all atoms within a small volume  $\delta V$  with position  $z$ .  $N$  is the number density of the atoms inside the cell. The factor  $e^{-ikz}$ , with  $k = \omega/c$  being the central wavevector of the optical field, has been included in Eq. (6.62) to ensure the slow variation of atomic operators as a function of position. Show that, for example,

$$[\hat{\sigma}_{aa}(z, t), \hat{\sigma}_{ba}(z', t)] = \frac{L}{NV} \delta(z - z') \hat{\sigma}_{ab}(z, t),\tag{6.63}$$

where  $V$  is the volume of the cell and  $L$  its length.

**Hint:** because our treatment is one-dimensional, write  $\delta V = A\delta z$ , where  $A$  is the area of the sample and  $\delta z$  is a short cell length interval. Calculate the commutator for finite  $\delta z$  and show that it approaches the delta function for  $\delta z \rightarrow 0$ .

**Note 6.12** Other position-dependent atomic operators are defined similarly to Eq. (6.62).

**Problem 6.30** We will now treat the light quantum-mechanically. Introduce the slowly-varying position-dependent optical operator

$$\hat{a}(z, t) = \sqrt{\frac{2\epsilon_0 V}{\hbar\omega}} \hat{E}^{(+)}(z, t) e^{-ikz+i\omega t} = \sum_j \hat{a}_j(t) e^{-i(k-k_j)z+i\omega t}, \quad (6.64)$$

where  $j$  indexes the plane-wave modes and the quantization volume is taken equal to the cell volume. Show that

$$[\hat{a}(z, t), \hat{a}^\dagger(z', t)] = L\delta(z - z') \quad (6.65)$$

**Hint:** Because our treatment is one-dimensional,  $k_j = (2\pi/L)j$ . For all relevant optical modes, one can assume  $\omega_j \approx \omega$ . Use the relation  $\sum_{j=-\infty}^{+\infty} e^{ijx} = 2\pi\delta(x)$ .

**Problem 6.31** Show that we can rewrite the propagation equation (1.19) for the slowly-varying envelope as follows:

$$[c\partial_z + \partial_t] \hat{a}(z, t) = igNV\hat{\sigma}_{ba}(z, t), \quad (6.66)$$

where

$$g = d\sqrt{\frac{\omega}{2\hbar\epsilon_0 V}} \quad (6.67)$$

is the coupling coefficient. Assume that the index of refraction is close to one.

**Hint:** Although (1.19) is written for the nonlinear polarization, it is also valid in the linear case provided that the linear dispersion is not already accounted for in the left-hand side (i.e.  $\omega/k = c$ ). In the case of EIT, the index of refraction is very close to 1 but the dispersion is significant (see Problem 1.11). The polarization in the right-hand side accounts for this dispersion.

**Problem 6.32** Reproduce result (6.66) using the Heisenberg equations of motion for the optical operators in the slowly-varying operator picture.

**Problem 6.33** Show that the evolution for the atomic states, including the spontaneous decay terms, can be written as

$$\partial_t \hat{\sigma}_{aa} = -ig\hat{a}^\dagger \hat{\sigma}_{ba} + ig\hat{a} \hat{\sigma}_{ab} - \Gamma \hat{\sigma}_{aa} \quad (6.68a)$$

$$\partial_t \hat{\sigma}_{ab} = -i\Delta \hat{\sigma}_{ab} - ig\hat{a}^\dagger (\hat{\sigma}_{bb} - \hat{\sigma}_{aa}) - (\Gamma/2) \hat{\sigma}_{ab} \quad (6.68b)$$

$$\partial_t \hat{\sigma}_{ba} = i\Delta \hat{\sigma}_{ba} + ig\hat{a} (\hat{\sigma}_{bb} - \hat{\sigma}_{aa}) - (\Gamma/2) \hat{\sigma}_{ba} \quad (6.68c)$$

$$\partial_t \hat{\sigma}_{bb} = -ig\hat{a} \hat{\sigma}_{ab} + ig\hat{a}^\dagger \hat{\sigma}_{ba} + \Gamma \hat{\sigma}_{aa} \quad (6.68d)$$

where all optical and atomic operators are understood as functions of space and time.

**Note 6.13** A set of equations fully describing the evolution of the atom-light system is called the *Maxwell-Bloch equations*. An example is Eqs. (6.66) and (6.68). In Maxwell-Bloch equations, the light can be treated either classically or quantum-mechanically; the atoms are always treated quantum-mechanically.

**Problem 6.34** Write the set of Maxwell-Bloch equations for an EIT system. Neglect both detunings. Treat the control field classically and the signal quantum-mechanically. The control field is externally controlled and depends on time only. For a weak signal field, the atoms remain in  $|b\rangle$  and thus we can write  $\hat{\sigma}_{bb} \equiv \hat{1}$  and neglect  $\hat{\sigma}_{aa}$ ,  $\hat{\sigma}_{cc}$  and  $\hat{\sigma}_{ac}$  because they are of the second order in the signal field. Show that the propagation equation for the signal field remains of form (6.66) and for the atomic transitions we have

$$\partial_t \hat{\sigma}_{ba} = ig\hat{a} + i\Omega_c \hat{\sigma}_{bc} - (\Gamma/2) \hat{\sigma}_{ba} \quad (6.69a)$$

$$\partial_t \hat{\sigma}_{bc} = i\Omega_c \hat{\sigma}_{ba} \quad (6.69b)$$

**Problem 6.35** Show that

$$\hat{\sigma}_{bc}(z, t) = -\frac{g}{\Omega_c} \hat{a}(z, t) \quad (6.70)$$

and thus the propagation equation takes the form

$$[c\partial_z + \partial_t] \hat{a}(z, t) = -\frac{g^2 NV}{\Omega_c} \partial_t \frac{\hat{a}(z, t)}{\Omega} \quad (6.71)$$

Use Eq. (1.20) to verify that the group velocity that obtains from the above is consistent with Eq. (6.47).

**Hint:** adiabatically eliminate  $\hat{\sigma}_{ba}$ .

**Note 6.14** When  $\Omega_c$  is constant, Eq. (6.71) describes the motion of the signal field to be with constant velocity  $v_{\text{gr}}$ . However, when the control field is varied, the motion becomes more complicated because  $\Omega$  is a part of the derivative in the right-hand side of Eq. (6.71). Then we can describe the motion of the pulse using so-called *dark-state polaritons*<sup>3</sup>:

$$\hat{\Psi}(z, t) = \frac{1}{\sqrt{\Omega_c^2 + g^2 NV}} [\Omega_c \hat{a}(z, t) - g NV \hat{\sigma}_{bc}(z, t)]. \quad (6.72)$$

**Problem 6.36** For the dark-state polariton:

a) Show that

$$[\hat{\Psi}(z, t), \hat{\Psi}^\dagger(z', t)] = L\delta(z - z') \quad (6.73)$$

if  $\hat{a}(z, t)$  and  $\hat{\sigma}_{bc}(z, t)$  are treated as independent operators.

b) Show that the motion of the dark-state polariton is described by

$$\left[ c \frac{\Omega_c^2}{g^2 NV + \Omega_c^2} \partial_z + \partial_t \right] \hat{\Psi}(z, t) = 0. \quad (6.74)$$

In other words, the polariton travels with velocity  $v_{\text{gr}}$  and can be arbitrarily decelerated, accelerated or stopped by varying the control field intensity.

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<sup>3</sup>M. Fleischhauer and M. D. Lukin, Phys. Rev. A **65**, 022314 (2002).

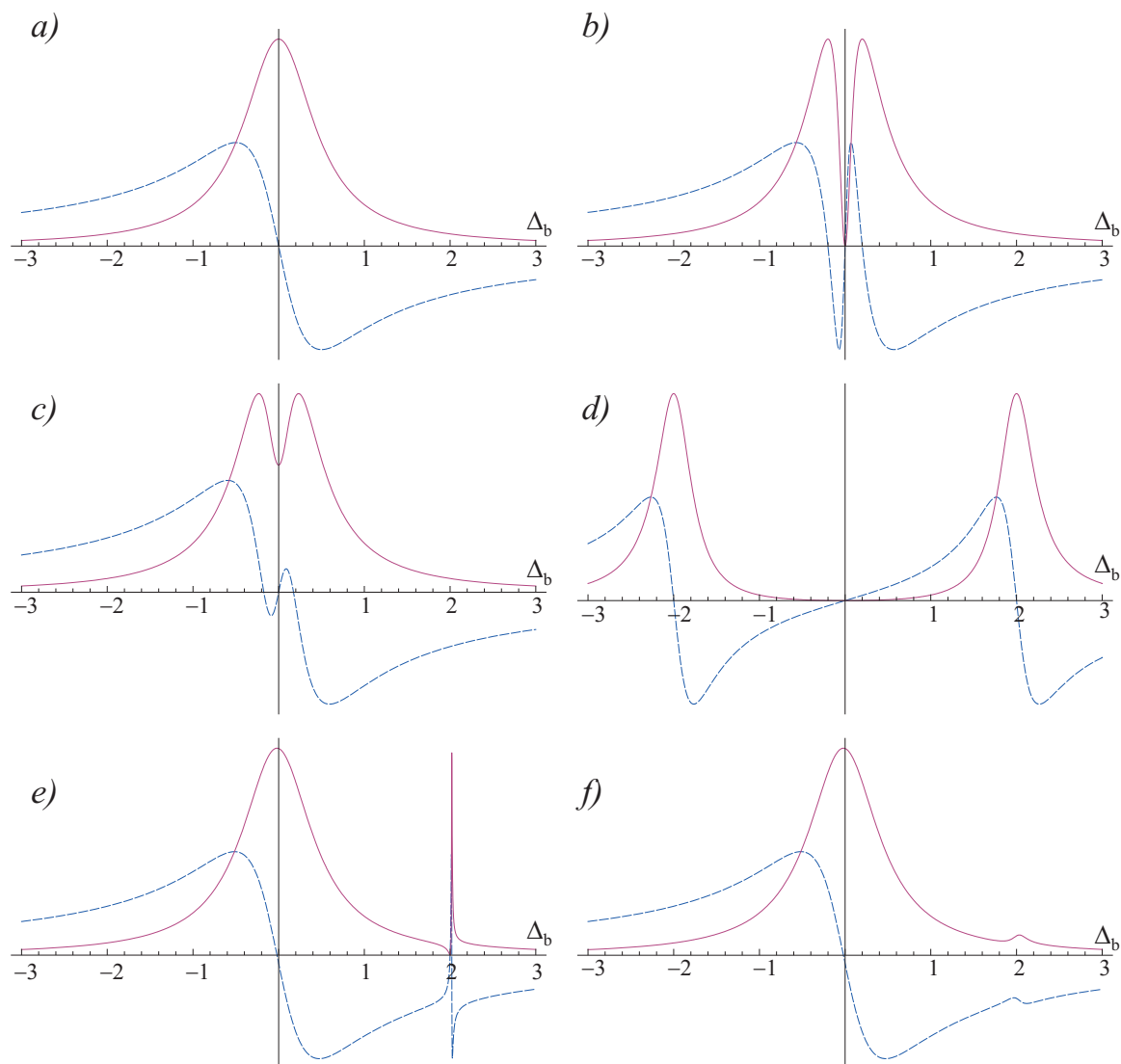


Figure 6.5: Real (dashed) and imaginary (solid) components of the susceptibility of the atomic gas with respect to the probe field under various EIT conditions. The horizontal axis is in units of  $\Gamma_1 + \Gamma_2$ . (a)  $\Omega_c = 0$  (no EIT). (b)  $\Omega_c = 0.2\Gamma$ ,  $\Delta_c = 0$ ,  $\gamma = 0$  (regular EIT line). (c)  $\Omega_c = 0.2\Gamma$ ,  $\Delta_c = 0$ ,  $\gamma = 0.1$  (EIT contrast reduced in the presence of ground state decoherence) (d)  $\Omega_c = 2\Gamma$ ,  $\Delta_c = 0$ ,  $\gamma = 0$  (large control field leads to significant ac Stark shift of the  $|a\rangle - |c\rangle$  transition and, so the probe field interacts with the dressed states - the phenomenon known as *Autler-Townes splitting*). (e)  $\Omega_c = 0.2\Gamma$ ,  $\Delta_c = 2\Gamma$ ,  $\gamma = 0$  (Raman absorption line). (f)  $\Omega_c = 0.2\Gamma$ ,  $\Delta_c = 2\Gamma$ ,  $\gamma = 0.1$  (the Raman line is extremely sensitive to decoherence).