Chapter 3

Fundamentals of the coherence theory

3.1 Temporal coherence

Consider an electromagnetic field $E(t)$ whose amplitude is not constant, but drifts randomly in the complex plane. Such a field can be treated as a stochastic process. The primary characteristic of a stochastic process is its *correlation function*

$$\Gamma(t,\tau) = \langle E^-(t)E^+(t+\tau)\rangle, \tag{3.1}$$

where

$$E^+(t) = \int_0^\infty E_F(\omega)e^{-i\omega t}\mathrm{d}\omega \tag{3.2}$$

is the positive-frequency part of the field, $E^-(t) = (E^+(t))^*$ and the averaging is done over all possible realizations of the process. We assume that the stochastic process associated with the field is *stationary*, i.e. its correlation function is time-independent[1]: $\Gamma(t,\tau) \equiv \Gamma(\tau)$. The quantity

$$g^{(1)}(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)} \tag{3.3}$$

is called the *degree of coherence* or *first-order coherence function.* The *coherence time* of the wave is defined as

$$\tau_c = \sqrt{\frac{\int_{-\infty}^{+\infty}\tau^2|g^{(1)}(\tau)|^2\mathrm{d}\tau}{\int_{-\infty}^{+\infty}|g^{(1)}(\tau)|^2\mathrm{d}\tau}}. \tag{3.4}$$

Problem 3.1 Show that

a) $g^{(1)}(\tau) = g^{(1)}(-\tau)^*$;

b) $|g^{(1)}(\tau)| \leq 1$;

Problem 3.2 Calculate the correlation function and the coherence time for the field $E(t) = \mathcal{E}(t)e^{-i\omega_0 t} + c.c.$ whose amplitude $\mathcal{E}(t)$ exhibits random jumps between values $+a$ and $-a$ (*random telegraph signal*). The probability for a jump to occur within time interval $\mathrm{d}t$ is given by $R\mathrm{d}t$, with $R \ll \omega_0$.
Hint: The number of jumps that may occur between moments t and $t+\tau$ follows Poissonian statistics (why?).

[1] This restricts our analysis to continuous fields and excludes pulsed fields.

Problem 3.3 A wave with temporal coherence function $\gamma(\tau)$ enters a fiber Mach-Zehnder interferometer (Fig. 3.1) with temporal path-length difference τ. The interference fringes are observed with a detector; the averaging time is much longer than the coherence time of the wave. Show that the visibility of the interference pattern is given by

$$\mathcal{V}(\tau) = |g^{(1)}(\tau)|. \tag{3.5}$$

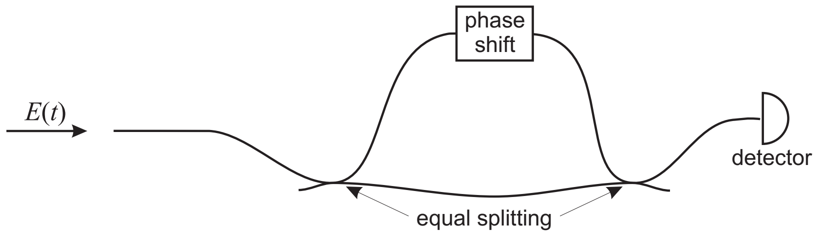

Figure 3.1: Mach-Zehnder fiber interferometer.

Problem 3.4 Verify that

$$\langle E_F^*(\omega) E_F(\omega') \rangle = \Gamma_F(\omega)\delta(\omega - \omega') \tag{3.6}$$

where $\Gamma_F(\omega)$ is the Fourier transform of the correlation function (3.1):

$$\Gamma_F(\omega) = \frac{1}{2\pi}\Gamma(\tau)e^{i\omega\tau}\mathrm{d}\tau. \tag{3.7}$$

Problem 3.5 Consider a bichromatic field $E(t) = E_1 e^{-i\omega_1 t} + E_2 e^{-i\omega_2 t}$.

a) Find the amplitude spectrum $E_F(\omega)$ as well as the correlation function $\Gamma(\tau)$ and its Fourier image $\Gamma_F(\omega)$ for this field.

b) Verify that the results of part (a) are inconsistent with Eq. (3.6) and explain why this is the case.

Problem 3.6 For an arbitrary stationary field $E(t)$, calculate the intensity I as the absolute value of the Poynting vector averaged over one optical period:

$$I(t) = 2nc\epsilon_0 E^-(t)E^+(t). \tag{3.8}$$

Show that

$$\langle I(t) \rangle = 2nc\epsilon_0 \int_0^\infty \Gamma_F(\omega)\mathrm{d}\omega. \tag{3.9}$$

Note 3.1 The result (3.9) means that the quantity $2nc\epsilon_0\Gamma_F(\omega)$ has the meaning of the spectral density of the electric field energy flow: the differential $2nc\epsilon_0\Gamma_F(\omega)\mathrm{d}\omega$ gives the intensity associated with the wave components whose frequencies lie in the interval between ω and $\omega + \mathrm{d}\omega$. The fact that the spectral power density is proportional to the Fourier transform of the correlation function is known as the *Wiener-Khinchine theorem*. It is of fundamental importance for any application that involves processing of oscillatory signal. For example, the electronic spectrum analyzer displays the spectral power of its input signal within the resolution bandwidth. This quantity is the Fourier transform of the correlation function of the input signal.

The Wiener-Khinchine theorem can also be used in the reverse way, for calculating the optical coherence from the spectral power density. An example is the following exercise.

Problem 3.7 The input to the interferometer of Fig. 3.1 is white light from an incandescent bulb that has passed through a spectral filter of width δ. Assuming the transmission of the filter to be given by $T(\omega) = e^{-(\omega-\omega_0)^2/2\delta^2}$,

a) calculate the interference visibility as a function of the path-length difference;

b) find the coherence time of the interferometer input.

Note 3.2 The result $\tau_c\delta \sim 1$ if the familiar time-energy uncertainty relation for electromagnetic waves.

3.2 Second-order coherence and thermal light

The *second-order coherence function* or *intensity correlation function* is defined as

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}, \tag{3.10}$$

where the instantaneous intensity is given by Eq. (3.8). This quantity is convenient to evaluate experimentally with a single photodiode connected to an oscilloscope which records the photocurrent as a function of time.

Problem 3.8 Show that

a) $g^{(2)}(\tau) \geq 0$;

b) $g^{(2)}(\tau) = g^{(2)}(-\tau)$;

c) $g^{(2)}(0) \geq 1$;

d) $g^{(2)}(\tau) \leq g^{(2)}(0)$;

Note 3.3 The last two properties of the second-order coherence do not remain valid if the light is treated quantum-mechanically. Experiential violations of these properties are viewed as evidence of nonclassical character of a light source.

Problem 3.9 Calculate $g^{(2)}(\tau)$ for perfectly coherent light.

Perfectly *white (chaotic)* light has energy spectrum $\Gamma_F(\omega)$ = const and hence the value of the electric field $E(t)$ at each moment in time is completely uncorrelated with the field at any other moments. Its spectral density is constant.

Perfectly white light is an abstraction. However, some of its properties are well approximated by *thermal* light emitted by a black body in a thermodynamic equilibrium. The energy spectrum of thermal light is given by Planck's formula

$$\Gamma_F(\omega, T) \propto \frac{\omega^3}{e^{\hbar\omega/kT} - 1}, \tag{3.11}$$

where T is the temperature of the source (Fig. 3.2).

Thermal light is obtained due to interference of radiation from multiple partially coherent sources of different frequencies, for example, excited atoms. According to the central limit theorem, this interference results in a Gaussian probability distribution of probabilities to observe a particular value of the field at each moment: $\mathrm{pr}[E(t)] = \exp[-(E(t)/E_0)^2]$.

Now let us suppose that thermal light is subjected to narrow spectral filtering with amplitude transmission $U_F(\omega)$ whose width δ is significantly less than the central frequency ω_0. Because the filter enacts a linear transformation we can write (*cf.* Problem 1.3):

$$E_F(\omega)_{\text{after filter}} = U_F(\omega)E_F(\omega)_{\text{before filter}} \tag{3.12}$$

and thus

$$E(t)_{\text{after filter}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(\tau)E_F(t-\tau)_{\text{before filter}}\mathrm{d}\tau, \tag{3.13}$$

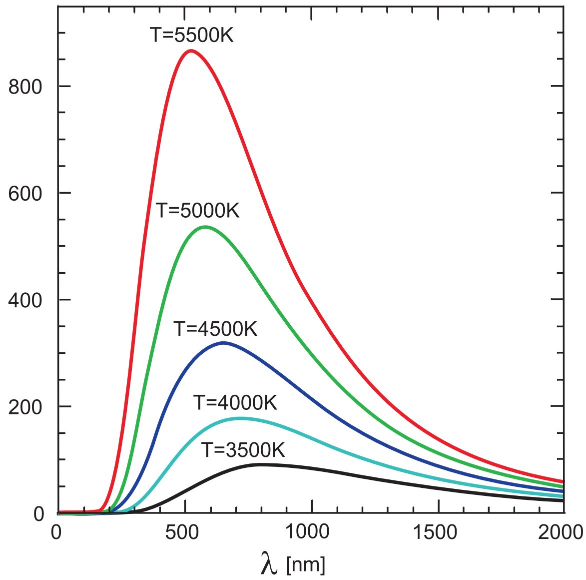

Figure 3.2: Thermal spectrum according to Planck's formula (from *Wikipedia*).

where $U(\tau)$ is the inverse Fourier transform of $U_F(\omega)$. Due to causality, $U(\tau)$ vanishes at $\tau < 0$. In other words, the field after the filter is a linear function of the field before the filter at all preceding moments.

The point of the story is that the left-hand side of Eq. (3.13) is a sum of Gaussian random variables, so it must be a Gaussian variable itself. In addition (see Problem 3.10), the filter introduces correlations between fields at different moments in time. Careful analysis shows that these correlations are also of Gaussian form. This is important because it allows us to apply the *Gaussian moment theorem (Isserlis theorem)*, according to which all ≥ 2 moments of a multivariate Gaussian distribution can be expressed through the second-order moments. In particular,

$$\langle E^-(t_1)E^-(t_2)E^+(t_3)E^+(t_4)\rangle = \langle E^-(t_1)E^+(t_3)\rangle\langle E^-(t_2)E^+(t_4)\rangle + \langle E^-(t_1)E^+(t_4)\rangle\langle E^-(t_2)E^+(t_3)\rangle. \tag{3.14}$$

Problem 3.10 Using Eq. (3.13), express the degree of coherence of the field after the filter through $U(\tau)$. Verify consistency with the Wiener-Khinchine theorem (**Hint:** Remember that $U_F(\omega)$ is the filter transmission for the amplitude, not intensity).

Problem 3.11 Show that the probability distribution of the *intensity* of the thermal field after the filter follows thermal statistics:

$$\mathrm{pr}(I) = \frac{1}{I_0}e^{-I/I_0}. \tag{3.15}$$

Express I_0 through $\Gamma(0)$.
Hint: After a narrow filter, the amplitude of $E^+(t)$ (and $E^-(t)$) is slow varying with respect to one optical period, so it can be assumed constant when using Eq. (3.8). The real and imaginary parts of this amplitude are *independent*, and both of them obey Gaussian statistics.

Problem 3.12 Show that for the thermal light after the filter,

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2 \tag{3.16}$$

Note 3.4 According to Eq. (3.16) the intensity correlation function of thermal light is typically of the form shown in Fig. 3.3(a). At the same time, the intensity must obey thermal statistics (3.15). A typical behavior of the thermal light intensity as a function of time is shown in Fig. 3.3(b). We

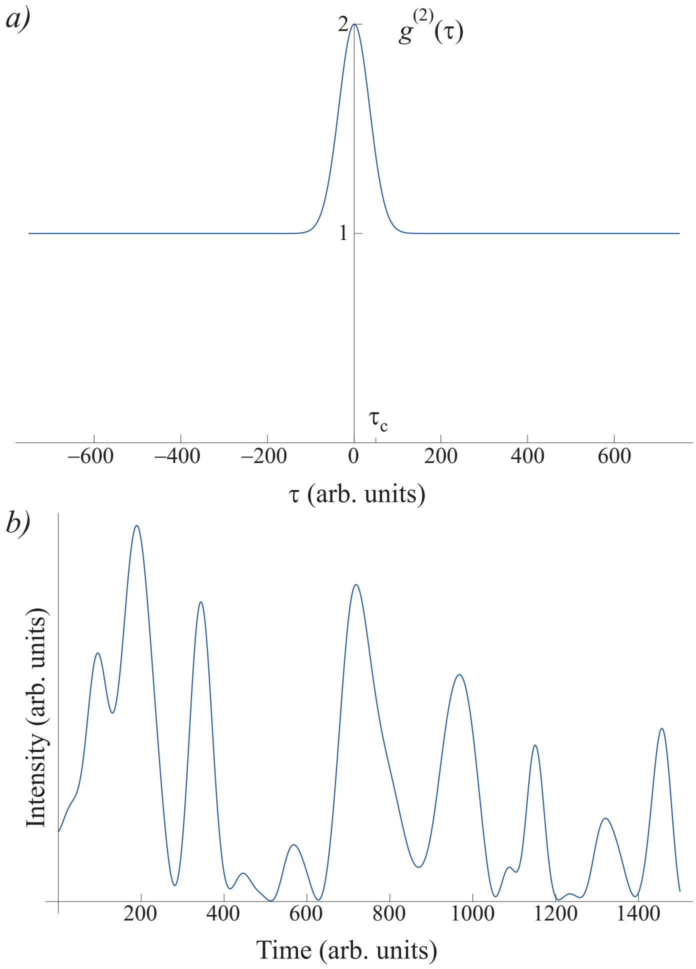

Figure 3.3: Thermal light. a) Intensity correlation function $g^{(2)}(\tau)$. b) Typical behavior of intensity as a function of time. The time scales are the same in (a) and (b).

see that the intensity fluctuates on a time scale of the coherence time. This is because the thermal light obtains as interference of light of multiple sources. Because these sources are incoherent, the interference pattern is *nonstationary*: its lifetime is restricted by the coherence time imposed by the filter.

Problem 3.13 Simulate thermal light on a computer using MatLab or Mathematica.

a) Generate two arrays of 100,000 random numbers from a Gaussian distribution to simulate perfectly chaotic light.

b) Subject these arrays to spectral filtering. To this end, convolve them with a Gaussian function of $1/e$ full width of 100 (the time scale is given by the array index). As we know, convolution corresponds to multiplication in the Fourier domain. Find the $1/e$ width of the corresponding frequency filter. The filtered arrays correspond to the real and imaginary parts of the complex field amplitude.

c) Calculate $g^{(1)}(\tau)$ from your data. Plot it for $-500 \leq \tau \leq 500$ together with the theoretical prediction obtained from the Wiener-Khinchine theorem.

d) Calculate the intensity $I(t)$. Plot $I(t)$ for any interval of length 3000. Plot the histogram of the intensity and verify its consistency with the thermal distribution in accordance with Problem 3.11.

e) Calculate $g^{(2)}(\tau)$ from your intensity data. Plot it for $-500 \leq \tau \leq 500$ together with the theoretical prediction obtained from Eq. (3.16).

Problem 3.14 The result shown in Fig. 3.3(a) appears surprising. Indeed, we expect the thermal light for $\tau < \tau_c$ to be indistinguishable from coherent light. Nevertheless, for thermal light we find $g^{(2)}(\tau < \tau_c) = 2$ while for coherent light $g^{(2)}(\tau) \equiv 1$ (Problem 3.9).

How can you explain this discrepancy? Confirm your explanation using the data from Problem 3.13.

3.3 Spatial coherence

The notion of temporal coherence can be straightforwardly generalized to the spatial domain. Instead of Eq. (3.1), for example, we can write

$$\Gamma(\vec{r}_1, \vec{r}_2, t, \tau) = \langle E^-(\vec{r}_1, \vec{r}_2, t) E^+(\vec{r}_1, \vec{r}_2, t+\tau) \rangle, \tag{3.17}$$

where $\vec{r}_1$ and $\vec{r}_2$ denote two spatial locations. For simplicity, when analyzing spatial coherence, we assume that the light is quasimonochromatic so the time dependence can be neglected. Assuming that the light is spatially uniform ("stationary" in space) we can assume that the correlation function depends only on the spatial separation $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$. Therefore, $\Gamma(\vec{r}_1, \vec{r}_2, t, \tau) \equiv \Gamma(\Delta\vec{r})$. The properties of the spatial correlation function are largely similar to those of its temporal counterpart.

Furthermore, it is convenient to assume that all components of the optical wave propagate at a small angle with respect to a specific direction, which we define as the z axis. In this *paraxial approximation*, we have $k_x, k_y \ll k_z$ and therefore the absolute value of the wavevector

$$k = c\sqrt{k_z^2 + k_x^2 + k_y^2} \approx k_z + \frac{k_x^2}{2k_z} + \frac{k_y^2}{2k_z} \tag{3.18}$$

is to the first order independent of its transverse component.

In the paraxial approximation, we can define the Fourier transform

$$E^+(\vec{r}) = e^{ikz} \int E_{\vec{k}_\perp} e^{i\vec{k}_\perp \cdot \vec{r}_\perp} \mathrm{d}^2\vec{k}_\perp, \tag{3.19}$$

where $\vec{r}_\perp = (x, y)$ and $\vec{k}_\perp = (k_x, k_y)$. Similarly to the Wiener-Khinchine theorem for temporal case, we have, under the condition of spatial uniformity,

$$E^*_{\vec{k}_\perp} E_{\vec{k}'_\perp} = \Gamma_{\vec{k}_\perp} \delta^2(\vec{k}_\perp - \vec{k}'_\perp) \tag{3.20}$$

where $\Gamma_{\vec{k}_\perp}$ is the Fourier image of the spatial correlation function and the energy flow density associated with the direction defined by $k_\perp$. This result is called the *van Cittert-Zernike theorem.* It can be used, for example, to measure diameters of stars.

Although each atom of a star emits a spherical wave, an observer on the Earth can assume it to be a plane wave (see Problem 3.18), so we can associate each point on the surface of the star with a specific wave vector $\vec{k}_\perp$. The angular size of the star determines the directional energy flow density $I_{\vec{k}_\perp} = 2c\epsilon_0 \Gamma_{\vec{k}_\perp}$, which is related, through the Fourier transform, to the experimentally measurable correlation function $\Gamma(\Delta\vec{r}_\perp)$ (Fig. 3.4(a)).

Because the light emitted by stars obeys thermal statistics, one can also measure their diameter by looking at intensity correlations rather than direct interferometry. For these we have, similarly to the previous section,

$$g^{(2)}(\Delta\vec{r}) = \frac{\langle I(\vec{r}) I(\vec{r} + \Delta\vec{r}) \rangle}{\langle I(\vec{r}) \rangle \langle I(\vec{r} + \Delta\vec{r}) \rangle} = 1 + |g^{(1)}(\Delta\vec{r})|^2. \tag{3.21}$$

The measurement is performed by observing correlations between photocurrents from two spatially separated detectors (*Hanbury Brown-Twiss experiment* — Fig. 3.4(b)). In fact, Hanbury Brown-Twiss measurements are preferred to classic interferometry because they are robust with respect to atmospheric fluctuations and easier to implement technically.

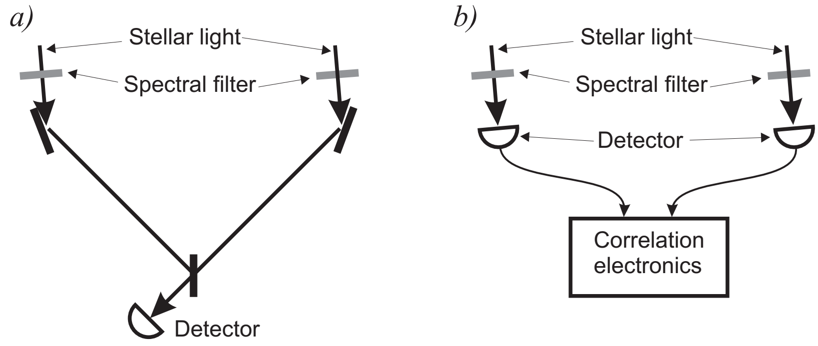

Figure 3.4: Measuring stellar diameter via (a) classic interferometry; (b) intensity (Hanbury Brown-Twiss) interferometry.

Problem 3.15 The course web site shows the image of a speckle pattern observed on an office wall when a green laser pointer of wavelength of wavelength $\lambda = 532$ nm was pointed at the opposite wall. The width of the image is $d = 12$ cm, the distance between walls $L = 4$ m.

a) Download the image and plot the histogram of the intensity (take only the green component into account). Explain possible reasons for deviations from the expected exponential shape.

b) Estimate the beam diameter of the laser pointer. **Hint:** Rather than calculating the intensity correlation function directly from the image data, use the Fast Fourier Transform to determine $I_{\vec{k}_\perp}$. Square its absolute value to find $\Gamma_{\vec{k}_\perp}$.

Problem 3.16 From the data shown in Fig. 3.5, estimate the diameter of Sirius given that its distance from the Earth is 8.6 light years. The wavelength can be assumed to be $\lambda = 320$ nm.

Problem 3.17 While we assumed the light to be monochromatic, a realistic monochromator has a finite linewidth.

a) What requirement does this bandwidth impose on the maximum allowed deviation of the position of the star from zenith in the measurement shown in Fig. 3.4(a)? Assume that the two mirrors are in the same horizontal plane and the two interferometer arms have the same length.

b) What requirement does this bandwidth impose on the response time of the photodetector in the Hanbury Brown-Twiss experiment? Explain your answer.

Problem 3.18 For Problem 3.16, estimate whether the plane wave approximation of a spherical wave emitted from each element of the star surface is valid. **Hint:** The difference between the wavefronts must be much less than a wavelength.

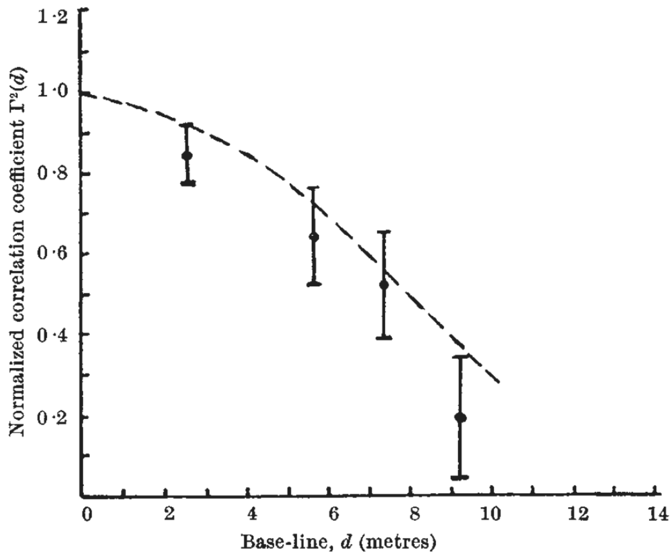

Figure 3.5: Experimental intensity correlation data for Hanbury Brown—Twiss interferometry of Sirius [From R. Hanbury Brown and R. Q. Twiss, Nature **178**, 1046 (1956)].